UNIT 1

PARTIAL DIFFERENTIATION (14 MARKS)

Topic Content:

1.1 Introduction to Derivative

1.2 Partial derivative (Two variables): Introduction,

Partial derivative of first order, second order and mixed order

1.3 Homogeneous Function

1.4 Euler's theorem on homogeneous function (Two variables)

1.5 Maxima and minima of function (Two variables)

1.6 Lagrange's method of undetermined multipliers with one constraint (Two variables)

Course Outcome: After completion of this course, students will be able to

CO1: Use partial differentiation concept to obtain optimal solution.

❖ Definition:

Let us consider a function z of the two variables x, y.

$i.e. z = f(x, y) \ldots \ldots . .1)$

Then the partial derivative of z w. r. to x is denoted as $\frac{\partial z}{\partial x}$

$$\frac{\partial z}{\partial x} = \frac{\partial f}{\partial x} = z_x = f_x$$

The partial derivative of z w. r. to y can be defined similarly & denoted as $\frac{\partial z}{\partial y}$

$$\frac{\partial z}{\partial y} = \frac{\partial f}{\partial y} = z_y = f_y$$

The partial derivatives of higher orders of the function (1) can also be calculated by successive partial differentiation, thus, we have

$$\frac{\partial^2 z}{\partial x^2} = \frac{\partial^2 f}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) = z_{xx} = f_{xx}$$

$$\frac{\partial^2 z}{\partial y^2} = \frac{\partial^2 f}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right) = z_{yy} = f_{yy}$$

$$\frac{\partial^2 z}{\partial y \partial x} = \frac{\partial^2 f}{\partial y \partial x} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) = z_{xy} = f_{xy}$$

$$\frac{\partial^2 z}{\partial x \partial y} = \frac{\partial^2 f}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) = z_{yx} = f_{yx}$$

- **Rules of Differentiation:**

Here u & v are functions of x.

1. $\frac{\partial}{\partial x}(u+v)=\frac{\partial}{\partial x}(u)+\frac{\partial}{\partial x}(v)$ [Sum Rule]
2. $\frac{\partial}{\partial x}(u-v)=\frac{\partial}{\partial x}(u)-\frac{\partial}{\partial x}(v)$ [Difference Rule]
3. $\frac{\partial}{\partial x}(k*u)=k\frac{\partial}{\partial x}(u)$ [Constant Multiple Rule]
4. $\frac{\partial}{\partial x}(u*v)=u\frac{\partial}{\partial x}(v)+v\frac{\partial}{\partial x}(u)$ [Product Rule]
5. $\frac{\partial}{\partial x}\left(\frac{u}{v}\right)=\frac{v\frac{\partial}{\partial x}(u)-u\frac{\partial}{\partial x}(v)}{v^2}$ [Quotient Rule]

DERIVATIVE

Some Important Formulae.

$$\frac{\partial}{\partial x} log[f(x,y)] = \frac{1}{f(x,y)}\frac{\partial}{\partial x}[f(x,y)]$$

$$\frac{\partial}{\partial x} sin[f(x,y)] = cos[f(x,y)]\frac{\partial}{\partial x}[f(x,y)]$$

$$\frac{\partial}{\partial x} tan^{-1}[f(x,y)] = \frac{1}{1+[f(x,y)]^2}\frac{\partial}{\partial x}[f(x,y)]$$

$$\frac{\partial}{\partial x}\left[\sqrt{f(x,y)}\right] = \frac{1}{2\sqrt{f(x,y)}}\frac{\partial}{\partial x}[f(x,y)]$$

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- **Examples:**

1. **Find the first & second partial derivative of z. $z = x^3 + y^3 - 3axy$ [S-23 4M]**

Sol: Given $z = x^3 + y^3 - 3axy$

Differentiate partially w. r. to x **[keeping y constant]**

$$\therefore \frac{\partial}{\partial x}(z) = \frac{\partial}{\partial x}[x^3 + y^3 - 3axy]$$

$$\therefore \frac{\partial z}{\partial x} = \frac{\partial}{\partial x}(x^3) + \frac{\partial}{\partial x}(y^3) - \frac{\partial}{\partial x}(3axy) \qquad \textbf{[Sum \& Difference rule]}$$

$$\therefore \frac{\partial z}{\partial x} = 3x^2 + 0 - 3ay \qquad \left\{\frac{\partial}{\partial x}[kx] = k, \quad k = 3ay\right\}$$

$$\boxed{\therefore \frac{\partial z}{\partial x} = 3x^2 - 3ay} \dots\dots\dots\dots 1)$$

Differentiate partially w. r. to y **[keeping x constant]**

$$\therefore \frac{\partial}{\partial y}(z) = \frac{\partial}{\partial y}[x^3 + y^3 - 3axy]$$

$$\therefore \frac{\partial z}{\partial y} = \frac{\partial}{\partial y}(x^3) + \frac{\partial}{\partial y}(y^3) - \frac{\partial}{\partial y}(3axy) \qquad \textbf{[Sum \& Difference rule]}$$

$$\therefore \frac{\partial z}{\partial y} = 0 + 3y^2 - 3ax \qquad \left\{\frac{\partial}{\partial y}[ky] = k, \quad k = 3ax\right\}$$

$$\boxed{\therefore \frac{\partial z}{\partial y} = 3y^2 - 3ax} \dots\dots\dots 2)$$

Differentiate 1) partially w. r. to x **[keeping y constant]**

$$\therefore \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial x}\right) = \frac{\partial}{\partial x}[3x^2 - 3ay]$$

$$\therefore \frac{\partial^2 z}{\partial x^2} = 3.2x - 0$$

$$\boxed{\therefore \frac{\partial^2 z}{\partial x^2} = 6x}$$

Differentiate 1) partially w. r. to y **[keeping x constant]**

$$\therefore \frac{\partial}{\partial y}\left(\frac{\partial z}{\partial x}\right) = \frac{\partial}{\partial y}[3x^2 - 3ay]$$

$$\therefore \frac{\partial^2 z}{\partial y \partial x} = 0 - 3a$$

$$\boxed{\therefore \frac{\partial^2 z}{\partial y \partial x} = -3a}$$

Differentiate 2) partially w. r. to x **[keeping y constant]**

$$\therefore \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial y}\right) = \frac{\partial}{\partial x}[3y^2 - 3ax]$$

$$\therefore \frac{\partial^2 z}{\partial x \partial y} = 0 - 3a$$

$$\boxed{\therefore \frac{\partial^2 z}{\partial x \partial y} = -3a}$$

Differentiate 2) partially w. r. to y **[keeping x constant]**

$$\therefore \frac{\partial}{\partial y}\left(\frac{\partial z}{\partial y}\right) = \frac{\partial}{\partial y}[3y^2 - 3ax]$$

$$\therefore \frac{\partial^2 z}{\partial y^2} = 3.2y - 0$$

$$\boxed{\therefore \frac{\partial^2 z}{\partial y^2} = -6y}$$

2. If $z = log(x^2 + y^2)$ then find $\frac{\partial z}{\partial x}$ & $\frac{\partial z}{\partial y}$

Sol: Given $z = log(x^2 + y^2)$

Differentiate z partially w. r. to x

$$\therefore \frac{\partial}{\partial x}(z) = \frac{\partial}{\partial x}\left[log(x^2 + y^2)\right]$$

$$\therefore \frac{\partial z}{\partial x} = \frac{1}{x^2 + y^2}\frac{\partial}{\partial x}(x^2 + y^2) \qquad \left\{\frac{\partial}{\partial x}log[f(x,y)] = \frac{1}{f(x,y)}\frac{\partial}{\partial x}[f(x,y)]\right\}$$

$$\therefore \frac{\partial z}{\partial x} = \frac{1}{x^2 + y^2}[2x + 0]$$

$$\boxed{\therefore \frac{\partial z}{\partial x} = \frac{2x}{x^2 + y^2}}$$

Differentiate z partially w. r. to y

$$\therefore \frac{\partial}{\partial y}(z) = \frac{\partial}{\partial y}[log(x^2 + y^2)]$$

$$\therefore \frac{\partial z}{\partial y} = \frac{1}{x^2 + y^2}\frac{\partial}{\partial y}(x^2 + y^2) \qquad \left\{\frac{\partial}{\partial y}log[f(x,y)] = \frac{1}{f(x,y)}\frac{\partial}{\partial y}[f(x,y)]\right\}$$

$$\therefore \frac{\partial z}{\partial y} = \frac{1}{x^2 + y^2}[0 + 2y]$$

$$\boxed{\therefore \frac{\partial z}{\partial y} = \frac{2y}{x^2 + y^2}}$$

3. If $z = 5x^4 + 2x^3y^2 - 3y$ find $\frac{\partial z}{\partial x}$ [S-24 2M]

Sol: Consider,

$z = 5x^4 + 2x^3y^2 - 3y$

Differentiate partially w. r. to x.

$$\therefore \frac{\partial}{\partial x}(z) = \frac{\partial}{\partial x}[5x^4 + 2x^3y^2 - 3y]$$

$$\therefore \frac{\partial z}{\partial x} = 5\frac{\partial}{\partial x}[x^4] + 2y^2\frac{\partial}{\partial x}[x^3] - 3y\frac{\partial}{\partial x}[1]$$

Using Formulae,

$$\frac{d}{dx}(x^n) = n.x^{n-1}, \frac{d}{dx}(k) = 0, k = constant$$

$$\therefore \frac{\partial z}{\partial x} = 5.4x^3 + 2y^2.3x^2 - 3y.0$$

$$\boxed{\therefore \frac{\partial z}{\partial x} = 20x^3 + 6x^2y^2}$$

4. If $z = e^{-t}sin\,\theta$ find $\frac{\partial^2 z}{\partial t \partial \theta}$ [S-24 2M]

Sol: Given,

$z = e^{-t}sin\,\theta$

Diff. partially w. r. to θ

$$\therefore \frac{\partial}{\partial \theta}(z) = \frac{\partial}{\partial \theta}[e^{-t}sin\,\theta]$$

$$\therefore \frac{\partial z}{\partial \theta} = e^{-t}\frac{\partial}{\partial \theta}[sin\,\theta]$$

$$\boxed{\therefore \frac{\partial z}{\partial \theta} = e^{-t}.\cos\theta}$$

Again diff. partially w. r. to t

$$\therefore \frac{\partial}{\partial t}\left[\frac{\partial z}{\partial \theta}\right] = \frac{\partial}{\partial t}[e^{-t}.\cos\theta]$$

$$\therefore \frac{\partial^2 z}{\partial t \partial \theta} = \cos\theta\frac{\partial}{\partial t}[e^{-t}]$$

Using formula. $\frac{d}{dx}(e^{ax}) = ae^{ax}, a = -1$

$$\therefore \frac{\partial^2 z}{\partial t \partial \theta} = \cos\theta \,. -e^{-t}$$

$$\boxed{\therefore \frac{\partial^2 z}{\partial t \partial \theta} = -e^{-t}.\cos\theta}$$

5. **If $f(x,y) = 3x + 4xy$ find $\frac{\partial f}{\partial x}$ [W-23 2M]**

Sol: Given, $f(x,y) = 3x + 4xy$

Diff. partially w. r. to x.

$$\therefore \frac{\partial}{\partial x}[f(x,y)] = \frac{\partial}{\partial x}[3x + 4xy]$$

$$\therefore \frac{\partial f}{\partial x} = 3\frac{\partial}{\partial x}[x] + 4y\frac{\partial}{\partial x}[x]$$

$$\therefore \frac{\partial f}{\partial x} = 3.1 + 4y.1$$

$$\boxed{\therefore \frac{\partial f}{\partial x} = 3 + 4y}$$

6. **If $f(x,y) = x^2y + \sin x + \cos y$ find $\frac{\partial^2 f}{\partial x \partial y}$ [W-23 2M]**

Sol: Given, $f(x,y) = x^2y + \sin x + \cos y$

Diff. partially w. r. to y.

$$\therefore \frac{\partial}{\partial y}[f(x,y)] = \frac{\partial}{\partial y}[x^2y + \sin x + \cos y]$$

$$\therefore \frac{\partial f}{\partial y} = x^2\frac{\partial}{\partial y}[y] + \frac{\partial}{\partial y}[\sin x] + \frac{\partial}{\partial y}[\cos y]$$

$$\therefore \frac{\partial f}{\partial y} = x^2.1 + 0 + (-\sin y)$$

$$\boxed{\therefore \frac{\partial f}{\partial y} = x^2 - \sin y}$$

Again differentiating partially w. r. to x.

$$\therefore \frac{\partial}{\partial x}\left[\frac{\partial f}{\partial y}\right] = \frac{\partial}{\partial x}[x^2 - \sin y]$$

$$\therefore \frac{\partial^2 f}{\partial x \partial y} = \frac{\partial}{\partial x}[x^2] - \frac{\partial}{\partial x}[\sin y]$$

$$\therefore \frac{\partial^2 f}{\partial x \partial y} = 2x - 0$$

$$\boxed{\therefore \frac{\partial^2 f}{\partial x \partial y} = 2x}$$

7. If $f(x,y)=x^3+y^3+6xy$ find $\frac{\partial f}{\partial x},\frac{\partial f}{\partial y}$ & $\frac{\partial^2 f}{\partial y\partial x}$ [W-23 4M]

Sol: Given, $f(x,y)=x^3+y^3+6xy \ldots\ldots 1)$

Diff. partially w. r. to x.

$$\therefore \frac{\partial}{\partial x}[f(x,y)]=\frac{\partial}{\partial x}[x^3+y^3+6xy]$$

$$\therefore \frac{\partial f}{\partial x}=\frac{\partial}{\partial x}[x^3]+\frac{\partial}{\partial x}[y^3]+6y\frac{\partial}{\partial x}[x]$$

$$\therefore \frac{\partial f}{\partial x}=3x^2+0+6y.1$$

$$\boxed{\therefore \boldsymbol{\frac{\partial f}{\partial x}=3x^2+6y}} \ldots\ldots\ldots \mathbf{2)}$$

Diff. partially w. r. to y.

$$\therefore \frac{\partial}{\partial y}[f(x,y)]=\frac{\partial}{\partial y}[x^3+y^3+6xy]$$

$$\therefore \frac{\partial f}{\partial y}=\frac{\partial}{\partial y}[x^3]+\frac{\partial}{\partial y}[y^3]+6x\frac{\partial}{\partial y}[y]$$

$$\therefore \frac{\partial f}{\partial y}=0+3y^2+6x.1$$

$$\boxed{\therefore \boldsymbol{\frac{\partial f}{\partial y}=3y^2+6x}} \ldots\ldots\ldots \mathbf{3)}$$

Diff. partially 2) w. r. to y.

$$\therefore \frac{\partial}{\partial y}\left[\frac{\partial f}{\partial x}\right]=\frac{\partial}{\partial y}[3x^2+6y]$$

$$\therefore \frac{\partial^2 f}{\partial y\partial x}=\frac{\partial}{\partial y}[3x^2]+6\frac{\partial}{\partial y}[y]$$

$$\therefore \frac{\partial^2 f}{\partial y\partial x}=0+6.1$$

$$\boxed{\therefore \boldsymbol{\frac{\partial^2 f}{\partial y\partial x}=6}}$$

8. If $f(x,y)=x\sin y+y\sin x+xy$ find $\frac{\partial f}{\partial x}$. [S-23 2M]

Sol: Given, $f(x,y)=x\sin y+y\sin x+xy$

Diff. partially w. r. to x.

$$\therefore \frac{\partial}{\partial x}[f(x,y)]=\frac{\partial}{\partial x}[x\sin y+y\sin x+xy]$$

$$\therefore \frac{\partial f}{\partial x}=\sin y\frac{\partial}{\partial x}[x]+y\frac{\partial}{\partial x}[\sin x]+y\frac{\partial}{\partial x}[x]$$

$$\therefore \frac{\partial f}{\partial x}=\sin y.1+y.\cos x+y.1$$

$$\boxed{\therefore \boldsymbol{\frac{\partial f}{\partial x}=\sin y+y.\cos x+y}}$$

9. If $\boldsymbol{f(x,y) = x^4 \sin y + y^4 \cos x + x^3}$ **find** $\frac{\partial^2 f}{\partial x \partial y}$ **[S-23 2M]**

Sol: Given, $f(x,y) = x^4 \sin y + y^4 \cos x + x^3$

Diff. partially w. r. to y.

$$\therefore \frac{\partial}{\partial y}[f(x,y)] = \frac{\partial}{\partial y}[x^4 \sin y + y^4 \cos x + x^3]$$

$$\therefore \frac{\partial f}{\partial y} = x^4 \frac{\partial}{\partial y}[\sin y] + \cos x \frac{\partial}{\partial y}[y^4] + \frac{\partial}{\partial y}[x^3]$$

$$\therefore \frac{\partial f}{\partial y} = x^4 . \cos y + \cos x . 4y^3 + 0$$

$$\boxed{\therefore \boldsymbol{\frac{\partial f}{\partial y} = x^4 \cos y + 4 \cos x\, y^3}}$$

Again differentiating partially w. r. to x.

$$\therefore \frac{\partial}{\partial x}\left[\frac{\partial f}{\partial y}\right] = \frac{\partial}{\partial x}[x^4 \cos y + 4 \cos x\, y^3]$$

$$\therefore \frac{\partial^2 f}{\partial x \partial y} = \cos y \frac{\partial}{\partial x}[x^4] + 4\, y^3 \frac{\partial}{\partial x}[\cos x]$$

$$\therefore \frac{\partial^2 f}{\partial x \partial y} = \cos y . 4x^3 + 4\, y^3(-\sin x)$$

$$\boxed{\therefore \boldsymbol{\frac{\partial^2 f}{\partial x \partial y} = 4x^3 \cos y - 4y^3 \sin x}}$$

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1.1 Introduction to Derivative

1.2 Partial derivative (Two variables): Introduction,

Partial derivative of first order, second order and mixed order

1.3 Homogeneous Function

1.4 Euler's theorem on homogeneous function (Two variables)

1.5 Maxima and minima of function (Two variables)

1.6 Lagrange's method of undetermined multipliers with one constraint (Two variables)

Course Outcome: After completion of this course, students will be able to

CO1: Use partial differentiation concept to obtain optimal solution.

❖ **EXTREMUM VALUES (MAXIMA & MINIMA):**

Consider the given function as $f(x,y) = 0$

STEP I: **Differentiate $f(x,y)$ to find $\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial^2 f}{\partial x^2}, \frac{\partial^2 f}{\partial x \partial y}, \frac{\partial^2 f}{\partial y^2}$**

STEP II: **Consider $\frac{\partial f}{\partial x} = 0$ & $\frac{\partial f}{\partial y} = 0$. Solve these equations to find values of x & y.**

Let (a,b) be the values of (x,y)

STEP III: **Find the values of $r = \frac{\partial^2 f}{\partial x^2}$, $s = \frac{\partial^2 f}{\partial x \partial y}$, $t = \frac{\partial^2 f}{\partial y^2}$ at point (a,b)**

STEP IV: **If $rt - s^2 > 0$ then function will have extremum value.**

a) **$r < 0$, then $f(x,y)$ has a maximum value at (a,b).**

b) **$r > 0$, then $f(x,y)$ has a minimum value at (a,b).**

STEP V: **If $rt - s^2 < 0$, then $f(x,y)$ has no extremum value at the point (a,b).**

STEP VI: **If $rt - s^2 = 0$, then then the case is doubtful and needs further investigation.**

NOTE: The point (a,b) which are the roots of $\frac{\partial f}{\partial x} = 0$ & $\frac{\partial f}{\partial y} = 0$ are called stationary points.

❖ **EXAMPLES:**

1. **Examine $f(x,y) = x^3 - y^2 - 3x$ for maximum and minimum values.**

Sol: Consider, $f(x,y) = x^3 - y^2 - 3x \dots\dots\dots 1)$

STEP I:

$$\frac{\partial f}{\partial x} = \frac{\partial}{\partial x}(x^3 - y^2 - 3x) = 3x^2 - 0 - 3 = 3x^2 - 3$$

$$\frac{\partial f}{\partial y} = \frac{\partial}{\partial y}(x^3 - y^2 - 3x) = 0 - 2y - 0 = -2y$$

$$r = \frac{\partial^2 f}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial}{\partial x}(3x^2 - 3) = 3.2x - 0 = 6x \dots\dots 2)$$

$$s = \frac{\partial^2 f}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial}{\partial x}(-2y) = 0 \dots\dots 3)$$

$t = \frac{\partial^2 f}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial}{\partial y}(-2y) = -2 \dots \dots .4)$

STEP II: Consider,

$\frac{\partial f}{\partial x} = 0$ & $\frac{\partial f}{\partial y} = 0$

$\therefore 3x^2 - 3 = 0$

$\therefore 3x^2 = 3$

$\therefore x = \frac{3}{3}$

$\therefore \boxed{x = \pm 1}$

$-2y = 0$

$y = \frac{0}{-2}$

$y = 0$

$\boxed{y = 0}$

The required points are $(1,0)$ & $(-1,0)$.

STEP III:

Stationary Points.	$(1,0)$	$(-1,0)$
$r = \frac{\partial^2 f}{\partial x^2} = 6x$	$r = 6(1) = 6$	$r = 6(-1) = -6$
$s = \frac{\partial^2 f}{\partial x \partial y} = 0$	$s = 0$	$s = 0$
$t = \frac{\partial^2 f}{\partial y^2} = -2$	$t = -2$	$t = -2$
$rt - s^2$	$rt - s^2 = (6)(-2) - (0)^2 = -12 < 0$	$rt - s^2 = (-6)(-2) - (0)^2 = 12 > 0$

STEP IV:

$rt - s^2 = (6)(-2) - (0)^2 = -12 - 0 = -12 < 0$

The given function will have no Extremum values at $(1,0)$.

$rt - s^2 = (-6)(-2) - (0)^2 = 12 - 0 = 12 > 0$

The given function will have Extremum values at $(-1,0)$.

Now, $r = -6 < 0$

The function $f(x,y)$ will have maximum value at point $(-1,0)$

Put $x = -1, y = 0$ in equation 1)

$\therefore f_{max} = (-1)^3 - (0)^2 - 3(-1)$

$\therefore f_{max} = -1 - 0 + 3$

$\boxed{\therefore f_{max} = 2}$

2. **Discuss the maxima & minima of the function $x^2 + y^2 + 6x + 12$**

Sol: Consider, $f(x,y) = x^2 + y^2 + 6x + 12 \dots \dots \dots 1)$

STEP I:

$\frac{\partial f}{\partial x} = \frac{\partial}{\partial x}(x^2 + y^2 + 6x + 12) = 2x + 0 + 6 + 0 = 2x + 6$

$\frac{\partial f}{\partial y} = \frac{\partial}{\partial y}(x^2 + y^2 + 6x + 12) = 0 + 2y + 0 + 0 = 2y$

$\frac{\partial^2 f}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial}{\partial x}(2x + 6) = 2 + 0 = 2$

$\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial}{\partial x}(2y) = 0$

$\frac{\partial^2 f}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial}{\partial y}(2y) = 2$

STEP II: Consider,

$\frac{\partial f}{\partial x} = 0$ & $\frac{\partial f}{\partial y} = 0$

$\therefore 2x + 6 = 0$ $\quad$ $2y = 0$

$\therefore 2x = -6$ $\quad$ $y = \frac{0}{2}$

$\therefore x = \frac{-6}{2}$ $\quad$ $y = 0$

$\therefore \boxed{x = -3}$ $\quad$ $\boxed{y = 0}$

The required point is $(-3,0)$.

STEP III:

Put $x = -3, y = 0$ in

$r = \frac{\partial^2 f}{\partial x^2} = 2,$

$s = \frac{\partial^2 f}{\partial x \partial y} = 0,$

$t = \frac{\partial^2 f}{\partial y^2} = 2$

STEP IV:

$rt - s^2 = (2)(2) - (0)^2 = 4 - 0 = 4 > 0$

The given function will have Extremum values.

Now, $r = 2 > 0$

The function $f(x, y)$ will have minimum value at point $(-3,0)$

Put $x = -3, y = 0$ in equation 1)

$\therefore f_{min} = (-3)^2 + (0)^2 + 6(-3) + 12$

$\therefore f_{min} = 9 + 0 - 18 + 12$

$\boxed{\therefore f_{min} = 3}$

3. **Find the local maxima & minima of function $f(x, y) = 2x^2 + 2xy + 2y^2 - 6x$ {w-23 4M}**

Sol: Given $f(x, y) = 2x^2 + 2xy + 2y^2 - 6x \ldots\ldots \mathbf{1})$

STEP I:

$\frac{\partial f}{\partial x} = \frac{\partial}{\partial x}(2x^2 + 2xy + 2y^2 - 6x) = 2.2x + 2y.1 + 0 - 6 = 4x + 2y - 6$

$\frac{\partial f}{\partial y} = \frac{\partial}{\partial y}(2x^2 + 2xy + 2y^2 - 6x) = 0 + 2x.1 + 2.2y - 0 = 2x + 4y$

$r = \frac{\partial^2 f}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial}{\partial x}(4x + 2y - 6) = 4 + 0 - 0 = 4 \ldots\ldots.2)$

$s = \frac{\partial^2 f}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial}{\partial x}(2x + 4y) = 2 + 0 = 2 \ldots\ldots.3)$

$t = \frac{\partial^2 f}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial}{\partial y}(2x + 4y) = 0 + 4 = 4 \ldots\ldots.4)$

STEP II: Consider,

$\frac{\partial f}{\partial x} = 0$ & $\frac{\partial f}{\partial y} = 0$

$\therefore 4x + 2y - 6 = 0$ $\quad$ $2x + 4y = 0$

$\therefore 2x + y - 3 = 0$ $\quad$ $x + 2y = 0$

$\therefore 2x + y = 3 \ldots\ldots.5)$ $\quad$ $x = -2y \ldots..6)$

Put $x = -2y$ in eq^n 5)

$\therefore 2(-2y) + y = 3$

$\therefore -4y + y = 3 \rightarrow -3y = 3$

$\therefore y = \frac{3}{-3} \rightarrow \boxed{\mathbf{y = -1}}$

Put $y = -1$ in eq^n 6)

$\therefore x = -2(-1) \rightarrow \boxed{\mathbf{x = 2}}$

The required point is (2,1).

STEP III:

Stationary Points.	$(2, -1)$
$r = \frac{\partial^2 f}{\partial x^2} = 4$	$r = 4$
$s = \frac{\partial^2 f}{\partial x \partial y} = 2$	$s = 2$
$t = \frac{\partial^2 f}{\partial y^2} = 4$	$t = 4$
$rt - s^2$	$rt - s^2 = (4)(4) - (2)^2 = 12 > 0$

STEP IV:

$rt - s^2 = 12 > 0$

The given function will have Extremum values at $(2, -1)$.

Now, $r = 4 > 0$

The function $f(x, y)$ will have minimum value at point $2, -1)$

Put $x = 2, y = 1$ in equation 1)

$\therefore f_{min} = 2(2)^2 + 2(2)(-1) + 2(-1)^2 - 6(2)$

$\therefore f_{min} = 8 - 4 + 2 - 12$

$\boxed{\therefore \boldsymbol{f_{min}} = \mathbf{-6}}$

4. **Examine $\boldsymbol{f\ (x, y) = x^3 + y^3 - 3axy}$ for maxima and minima where $\boldsymbol{a > 0}$.**

Sol: Given that $f\ (x, y) = x^3 + y^3 - 3axy \ldots \ldots .1)$

STEP I:

$$\frac{\partial f}{\partial x} = \frac{\partial}{\partial x}(x^3 + y^3 - 3axy) = 3x^2 + 0 - 3ay.1 = 3x^2 - 3ay$$

$$\frac{\partial f}{\partial y} = \frac{\partial}{\partial y}(x^3 + y^3 - 3axy) = 0 + 3y^2 - 3ax.1 = 3y^2 - 3ax$$

$$\frac{\partial^2 f}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial}{\partial x}(3x^2 - 3ay) = 3.2x - 0 = 6x$$

$$\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial}{\partial x}(3y^2 - 3ax) = 0 - 3a.1 = -3a$$

$$\frac{\partial^2 f}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial}{\partial y}(3y^2 - 3ax) = 3.2y - 0 = 6y$$

STEP II: Consider,

$\boldsymbol{\frac{\partial f}{\partial x} = 0}$ & $\boldsymbol{\frac{\partial f}{\partial y} = 0}$

$\therefore 3x^2 - 3ay = 0$ & $3y^2 - 3ax = 0$

$\therefore 3x^2 = 3ay$ & $3y^2 = 3ax$

$\therefore \frac{3x^2}{3a} = y$ & $y^2 = \frac{3ax}{3}$

$\therefore y = \frac{x^2}{a} \ldots\ldots 2)$ & $y^2 = ax \ldots\ldots 3)$

Putting the value of y from (2) in (3), we get.

$\therefore \left(\frac{x^2}{a}\right)^2 = ax \Rightarrow \frac{x^4}{a^2} = ax \Rightarrow x^4 = a^3x \Rightarrow x^4 - a^3x = 0$

$\therefore x(x^3 - a^3) = 0$

$\therefore x = 0, x^3 - a^3 = 0$

$\therefore x = 0, x^3 = a^3$

$\boxed{\therefore \boldsymbol{x = 0, x = a}}$

Putting $x = 0$ in (2), we get $y = \frac{(0)^2}{a} = 0$,

Putting $x = a$ in (2), we get $y = \frac{(a)^2}{a} = a$,

The required points are $(0,0), (a, a)$

STEP III:

Stationary Points.	$(0,0)$	(a, a)
$r = \frac{\partial^2 f}{\partial x^2} = 6x$	$r = 6(0) = 0$	$r = 6(a) = 6a$
$s = \frac{\partial^2 f}{\partial x \partial y} = -3a$	$s = -3a$	$s = -3a$
$t = \frac{\partial^2 f}{\partial y^2} = 6y$	$t = 6(0) = 0$	$t = 6(a) = 6a$
$rt - s^2$	$rt - s^2 = (0)(0) - (-3a)^2 = -9a^2 < 0$	$rt - s^2 = (6a)(6a) - (-3a)^2$ $= 36a^2 - 9a^2$ $= 27a^2 > 0$

STEP IV:

At $(0,0)$ there is no extremum value, since $rt - s^2 < 0$

At (a, a), there is extremum value, $rt - s^2 > 0$ & $r > 0$

The function $f\,(x, y)$ will have minimum value at point (a, a).

Put $x = a, y = a$ in equation 1)

$\therefore f\,(x, y)_{min} = (a)^3 + (a)^3 - 3a(a)(a) = 2a^3 - 3a^3$

$\boxed{\therefore \boldsymbol{f\,(x, y)_{min} = -a^3}}$

5. **Discuss the maxima & minima for the function $x^2 + y^2 + (30 - x - y)^2$**

Sol: Consider, $f(x, y) = x^2 + y^2 + (30 - x - y)^2 \ldots\ldots..1)$

STEP I:

$$\frac{\partial f}{\partial x} = \frac{\partial}{\partial x}(x^2 + y^2 + (30 - x - y)^2) = 2x + 0 + 2(30 - x - y)(0 - 1 - 0) = 2x - 60 + 2x + 2y$$
$$= 4x + 2y - 60$$

$$\frac{\partial f}{\partial y} = \frac{\partial}{\partial y}(x^2 + y^2 + (30 - x - y)^2) = 0 + 2y + 2(30 - x - y)(0 - 0 - 1) = 2y - 60 + 2x + 2y$$
$$= 2x + 4y - 60$$

$$\frac{\partial^2 f}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial}{\partial x}(4x + 2y - 60) = 4 + 0 - 0 = 4$$

$$\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial}{\partial x}(2x + 4y - 60) = 2 + 0 - 0 = 2$$

$$\frac{\partial^2 f}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial}{\partial y}(2x + 4y - 60) = 0 + 4 - 0 = 4$$

STEP II: Consider,

$\frac{\partial f}{\partial x} = 0$ & $\frac{\partial f}{\partial y} = 0$

$\therefore 4x + 2y - 60 = 0$ & $2x + 4y - 60 = 0$

$\therefore 4x + 2y = 60$ & $2x + 4y = 60$

Divide by 2.

$\therefore 2x + y = 30 \Rightarrow y = 30 - 2x \ldots\ldots\ldots 2)$ & $x + 2y = 30 \ldots\ldots\ldots 3)$

Put $y = 30 - 2x$ in equation 3)

$\therefore x + 2(30 - 2x) = 30$

$\therefore x + 60 - 4x = 30$

$\therefore -3x = 30 - 60 = -30$

$\therefore x = \frac{-30}{-3} \Rightarrow \boxed{x = 10}$

Put $x = 10$ in equation 2)

$\therefore y = 30 - 2(10) = 30 - 20 \Rightarrow \boxed{y = 10}$

The required point is (10,10)

STEP III:

Stationary Points.	$(10,10)$
$r = \frac{\partial^2 f}{\partial x^2} = 4$	$r = 4 > 0$
$s = \frac{\partial^2 f}{\partial x \partial y} = 2$	$s = 2$
$t = \frac{\partial^2 f}{\partial y^2} = 4$	$t = 4$
$rt - s^2$	$rt - s^2 = (4)(4) - (2)^2 = 16 - 4 = 12 > 0$

STEP IV:

At (10,10), there is extremum value, $rt - s^2 > 0 \,\&\, r > 0$

The function $f\,(x, y)$ will have minimum value at point (a, a).

Put $x = 10, y = 10$ in equation 1)

$\therefore f\,(x, y)_{min} = (10)^2 + (10)^2 + (30 - 10 - 10)^2$

$\therefore f\,(x, y)_{min} = 100 + 100 + 100$

$\boxed{\therefore f\,(x, y)_{min} = 300}$

6. **A rectangular box, open at the top, is to have a volume of 32 c.c. Find the dimensions of the box requiring least material for its construction.**

Sol: Let x, y, z be the length, breadth and height of the box. Given volume of the box is 32 cc.

$\Rightarrow xyz = 32 \ldots\ldots 1), \quad x, y, z > 0$

We want to minimize the amount of material for its construction.

i.e., surface area of the box is to be minimized.

Surface area $S = xy + 2xz + 2yz \ldots\ldots 2)$ [top is open]

From 1) $xyz = 32 \Rightarrow z = \frac{32}{xy}$

Put $z = \frac{32}{xy}$ in equation 2)

$\therefore S = xy + 2x\left(\frac{32}{xy}\right) + 2y\left(\frac{32}{xy}\right)$

$\therefore S = xy + \frac{64}{y} + \frac{64}{x}$

STEP I:

$\frac{\partial S}{\partial x} = \frac{\partial}{\partial x}\left(xy + \frac{64}{y} + \frac{64}{x}\right) = y.1 + 0 + 64.\frac{-1}{x^2} = y - \frac{64}{x^2}$

$\frac{\partial S}{\partial y} = \frac{\partial}{\partial y}\left(xy + \frac{64}{y} + \frac{64}{x}\right) = x.1 + 64.\frac{-1}{y^2} = x - \frac{64}{y^2}$

$\frac{\partial^2 S}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial S}{\partial x}\right) = \frac{\partial}{\partial x}\left(y - \frac{64}{x^2}\right) = 0 - 64.\frac{-2}{x^3} = \frac{128}{x^3}$ $\left\{\because \frac{\partial}{\partial x}\left(\frac{1}{x^2}\right) = \frac{-2}{x^3}\right\}$

$\frac{\partial^2 S}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial S}{\partial y}\right) = \frac{\partial}{\partial x}\left(x - \frac{64}{y^2}\right) = 1 - 0 = 1$

$\frac{\partial^2 S}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial S}{\partial y}\right) = \frac{\partial}{\partial y}\left(x - \frac{64}{y^2}\right) = 0 - 64.\frac{-2}{y^3} = \frac{128}{y^3}$

STEP II: Consider,

$\frac{\partial S}{\partial x} = 0$ & $\frac{\partial S}{\partial y} = 0$

$\therefore y - \frac{64}{x^2} = 0$ & $x - \frac{64}{y^2} = 0$

$\therefore y = \frac{64}{x^2} \ldots\ldots 2)$ & $x = \frac{64}{y^2} \ldots\ldots 3)$

Put $y = \frac{64}{x^2}$ in equation 3)

$\therefore x = \frac{64}{\left(\frac{64}{x^2}\right)^2} \Rightarrow x = \frac{64}{\frac{(64)^2}{x^4}} \Rightarrow x = \frac{x^4}{64} \Rightarrow 64 = \frac{x^4}{x}$

$\Rightarrow 64 = x^3$

Taking cube root on both sides.

$\boxed{\therefore 4 = x}$

Put $x = 4$ in equation 2)

$\therefore y = \frac{64}{4^2} = \frac{64}{16}$

$\boxed{\therefore y = 4}$

The required point is $(4,4)$.

STEP III:

Stationary Points.	$(4,4)$
$r = \frac{\partial^2 S}{\partial x^2} = \frac{128}{x^3}$	$r = \frac{128}{4^3} = 2 > 0$
$s = \frac{\partial^2 S}{\partial x \partial y} = 1$	$s = 1$
$t = \frac{\partial^2 S}{\partial y^2} = \frac{128}{y^3}$	$t = \frac{128}{4^3} = 2$
$rt - s^2$	$rt - s^2 = (2)(2) - (1)^2 = 4 - 1 = 3 > 0$

STEP IV:

At $(4,4)$, there is extremum value, $rt - s^2 > 0$ and $r > 0$

The function $S(x, y)$ will have minimum value at point $(4,4)$

Put $x = 4, y = 4$ in equation 1)

$\therefore (4)(4)z = 32 \Rightarrow 16z = 32 \Rightarrow z = \frac{32}{16} = 2$

∴ Dimensions of the box are $\boxed{x = 4\, cms, y = 4\, cms\ and\ z = 2\, cms.}$

Homework:

1. **Examine $f(x, y) = x^3 + y^3 - 12x - 3y + 20$ for its extreme value.**
   Ans: maximum value at $(-2, -1)$ is 38.
2. **Find the absolute maximum and minimum values of**

$$f\,(x, y) = 2 + 2x + 2y - x^2 - y^2$$

   Ans: Maximum value at $(1, 1)$ is 4

UNIT 1

PARTIAL DIFFERENTIATION (14 MARKS)

Topic Content:

1.1 Introduction to Derivative

1.2 Partial derivative (Two variables): Introduction,

Partial derivative of first order, second order and mixed order

1.3 Homogeneous Function

1.4 Euler's theorem on homogeneous function (Two variables)

1.5 Maxima and minima of function (Two variables)

1.6 Lagrange's method of undetermined multipliers with one constraint (Two variables)

Course Outcome: After completion of this course, students will be able to

CO1: Use partial differentiation concept to obtain optimal solution.

- **Lagrange's method of undetermined multipliers:**

Let $f(x, y)$ be the function whose extreme values are to be found subject to the restriction

$\phi\,(x, y) = 0$

STEP I: **Construct the auxiliary function, $F(x, y) = f(x, y) + \lambda.\phi f(x, y)$**

Where λ is called Lagrange's multiplier.

STEP II: **Find $\frac{\partial F}{\partial x}, \frac{\partial F}{\partial y}$**

STEP III: **Consider $\frac{\partial F}{\partial x} = 0, \frac{\partial F}{\partial y} = 0$ & solve these equations to find value of x, y. (Stationary Points)**

STEP IV: **Put values of x, y in $f(x, y)$ to find extremum value.**

- **EXAMPLE:**

1. **Find the extreme value of $f(x, y) = xy$ subject to condition $x + y = 10$ using Lagrange method of undetermined multipliers. [S-23 4M]**

Sol: Given, $f(x, y) = xy \ldots\ldots\ldots.1)$ & $\phi(x, y) = x + y - 10 = 0 \ldots\ldots\ldots.2)$

The auxiliary equation is,

$F(x, y) = f(x, y) + \lambda.\phi f(x, y)$

$\therefore F = xy + \lambda[x + y - 10]$

Diff. F partially w. r. to x.

$$\therefore \frac{\partial}{\partial x}(F) = \frac{\partial}{\partial x}\{xy + \lambda[x + y - 10]\}$$

$$\therefore \frac{\partial F}{\partial x} = y.1 + \lambda[1 + 0 - 0]$$

$$\boxed{\therefore \frac{\partial F}{\partial x} = y + \lambda}$$

Diff. F partially w. r. to y.

$$\therefore \frac{\partial}{\partial y}(F) = \frac{\partial}{\partial y}\{xy + \lambda[x + y - 10]\}$$

$$\therefore \frac{\partial F}{\partial y} = x.1 + \lambda[0 + 1 - 0]$$

$$\therefore \frac{\partial F}{\partial y} = x + \lambda$$

Now,

$$\frac{\partial F}{\partial x} = 0 \qquad \frac{\partial F}{\partial y} = 0$$

$$\therefore y + \lambda = 0 \qquad \therefore x + \lambda = 0$$

$$\therefore y = -\lambda \qquad \therefore x = -\lambda$$

$\therefore y = x$ ....3)

Put $y = x$ in equation 2)

$$\therefore x + x - 10 = 0$$

$$\therefore 2x - 10 = 0 \rightarrow 2x = 10 \rightarrow x = \frac{10}{2} \rightarrow x = 5$$

Put $x = 5$ in equation 3)

$\therefore y = 5$

The required point is (5,5)

Put (5,5) $\rightarrow x = 5, y = 5$ in equation 1)

$f(x,y) = xy \Longrightarrow f = (5)(5)$

$\therefore f_{max} = 25$

2. Examine $f(x,y) = xy$ subjected to constraint $g(x,y) = 4x^2 + y^2 = 8$ for maximum & minimum value. [w_23, S-24 4M]

Sol: Given, $f(x,y) = xy \ldots\ldots\ldots 1)$ & $g(x,y) = 4x^2 + y^2 - 8 = 0 \ldots\ldots\ldots 2)$

The auxiliary equation is,

$F(x,y) = f(x,y) + \lambda.\phi f(x,y)$

$$\therefore F = xy + \lambda[4x^2 + y^2 - 8]$$

Diff. F partially w. r. to x.

$$\therefore \frac{\partial}{\partial x}(F) = \frac{\partial}{\partial x}\{xy + \lambda[4x^2 + y^2 - 8]\}$$

$$\therefore \frac{\partial F}{\partial x} = y.1 + \lambda[4.2x + 0 - 0]$$

$$\therefore \frac{\partial F}{\partial x} = y + 8x\lambda$$

Diff. F partially w. r. to y.

$$\therefore \frac{\partial}{\partial y}(F) = \frac{\partial}{\partial y}\{xy + \lambda[4x^2 + y^2 - 8]\}$$

$$\therefore \frac{\partial F}{\partial y} = x.1 + \lambda[0 + 2y - 0]$$

$$\therefore \frac{\partial F}{\partial y} = x + 2y\lambda$$

Now,

$$\frac{\partial F}{\partial x} = 0$$

$$\therefore y + 8x\lambda = 0$$

$$\therefore y = -8x\lambda$$

$$\therefore \frac{y}{8x} = -\lambda$$

$$\frac{\partial F}{\partial y} = 0$$

$$\therefore x + 2y\lambda = 0$$

$$\therefore x = -2y\lambda$$

$$\therefore \frac{x}{2y} = -\lambda$$

$$\therefore \frac{y}{8x} = \frac{x}{2y}$$

$$\therefore 2y^2 = 8x^2$$

$$\therefore y^2 = 4x^2$$

$$\therefore y = +2x \; \ldots\ldots 3) \;\&\; \therefore y = -2x \; \ldots\ldots 4)$$

Put $y = 2x$ in equation 2)

$$\therefore 4x^2 + 4x^2 - 8 = 0$$

$$\therefore 8x^2 = 8 \rightarrow x^2 = \frac{8}{8} \rightarrow x = 1 \;\&\; x = -1$$

Put $x = 1$ in equation 3) & 4)

$\therefore y = 2(1) = 2 \;\&\; y = -2(1) = -2$

The required points are $(1,2), (1,-2)$

Put $x = -1$ in equation 3) & 4)

$\therefore y = 2(-1) = -2 \;\&\; y = -2(-1) = 2$

The required points are $(-1,-2), (-1,2)$

Put $(1,2) \rightarrow x = 1, y = 2$ in equation 1)

$f = (1)(2) = 2$

Put $(1,-2) \rightarrow x = 1, y = -2$ in equation 1)

$f = (1)(-2) = -2$

Put $(-1,-2) \rightarrow x = -1, y = -2$ in equation 1)

$f = (-1)(-2) = 2$

Put $(-1,2) \rightarrow x = -1, y = 2$ in equation 1)

$f = (-1)(2) = -2$

Function f has maximum value 2 at the points $(1,2)$ & $(-1,-2)$

Function f has minimum value 2 at the points $(1,-2)$ & $(-1,2)$

3. Find maximum & minimum value of function $f(x,y) = 3x + 4y$ on circle $x^2 + y^2 = 1$ using method of Lagrange's multiplier.

Sol: Given, $f(x,y) = 3x + 4y \ldots\ldots\ldots 1)$ & $\phi(x,y) = \boldsymbol{x^2 + y^2 - 1 = 0} \ldots\ldots\ldots \mathbf{2}$)

The auxiliary equation is,

$$F(x,y) = f(x,y) + \lambda.\phi f(x,y)$$

$$\therefore F = 3x + 4y + \lambda[\boldsymbol{x^2 + y^2 - 1}]$$

Diff. F partially w. r. to x.

$$\therefore \frac{\partial}{\partial x}(F) = \frac{\partial}{\partial x}\{3x + 4y + \lambda[\boldsymbol{x^2 + y^2 - 1}]\}$$

$$\therefore \frac{\partial F}{\partial x} = 3.1 + 0 + \lambda[2x + 0 - 0]$$

$$\boxed{\therefore \frac{\partial F}{\partial x} = 3 + 2x\lambda}$$

Diff. F partially w. r. to y.

$$\therefore \frac{\partial}{\partial y}(F) = \frac{\partial}{\partial y}\{3x + 4y + \lambda[\boldsymbol{x^2 + y^2 - 1}]\}$$

$$\therefore \frac{\partial F}{\partial y} = 0 + 4.1 + \lambda[0 + 2y - 0]$$

$$\boxed{\therefore \frac{\partial F}{\partial y} = 4 + 2y\lambda}$$

Now,

$$\frac{\partial F}{\partial x} = 0$$

$$\therefore 3 + 2x\lambda = 0$$

$$\therefore 3 = -2x\lambda$$

$$\therefore \frac{3}{2x} = -\lambda$$

$$\frac{\partial F}{\partial y} = 0$$

$$\therefore 4 + 2y\lambda = 0$$

$$\therefore 4 = -2y\lambda$$

$$\therefore \frac{4}{2y} = -\lambda$$

$$\therefore \frac{3}{2x} = \frac{4}{2y}$$

$$\therefore 6y = 8x$$

$$\therefore y = \frac{8}{6}x$$

$$\boxed{\therefore y = \frac{4}{3}x} \ldots\ldots\ldots 3)$$

Put $y = \frac{4}{3}x$ in equation 2)

$$\therefore x^2 + \left(\frac{4x}{3}\right)^2 - 1 = 0$$

$$\therefore x^2 + \frac{16x^2}{9} = 1$$

$$\therefore \frac{9x^2 + 16x^2}{9} = 1$$

$\therefore 25x^2 = 9$

$\therefore x^2 = \frac{9}{25}$

Taking square root.

$\boldsymbol{\therefore x = \frac{3}{5} \ \& \ x = -\frac{3}{5}}$

Put $x = \frac{3}{5}$ in equation 3)

$\therefore y = \frac{4}{3} \times \frac{3}{5}$

$\boldsymbol{\therefore y = \frac{4}{5}}$

The required point is $\left(\frac{3}{5}, \frac{4}{5}\right)$

Put $x = -\frac{3}{5}$ in equation 3)

$\therefore y = \frac{4}{3} \times -\frac{3}{5}$

$\boldsymbol{\therefore y = -\frac{4}{5}}$

The required point is $\left(-\frac{3}{5}, -\frac{4}{5}\right)$

Put $\left(\frac{3}{5}, \frac{4}{5}\right) \rightarrow x = \frac{3}{5}, y = \frac{4}{5}$ in equation 1)

$f(x, y) = 3x + 4y$

$\Longrightarrow f = 3\left(\frac{3}{5}\right) + 4\left(\frac{4}{5}\right)$

$\boldsymbol{\therefore f_{max} = 5}$

Put $\left(-\frac{3}{5}, -\frac{4}{5}\right) \rightarrow x = -\frac{3}{5}, y = -\frac{4}{5}$ in equation 1)

$f(x, y) = 3x + 4y$

$\Longrightarrow f = 3\left(-\frac{3}{5}\right) + 4\left(-\frac{4}{5}\right)$

$\boldsymbol{\therefore f_{min} = -5}$

Function f has maximum value 5 at the points $\left(\frac{3}{5}, \frac{4}{5}\right)$

Function f has minimum value -5 at the points $\left(-\frac{3}{5}, -\frac{4}{5}\right)$

UNIT 1

PARTIAL DIFFERENTIATION (14 MARKS)

❖ **Topic Content:**

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1.3 Homogeneous Function

1.4 Euler's theorem on homogeneous function (Two variables)

1.5 Maxima and minima of function (Two variables)

1.6 Lagrange's method of undetermined multipliers with one constraint (Two variables)

Course Outcome: **After completion of this course, students will be able to**

CO1: Use partial differentiation concept to obtain optimal solution.

❖ **Definition:**

An expression of the form,

$$f(x,y) = a_0x^n + a_1x^{n-1}y + a_2x^{n-2}y^2 + a_3x^{n-3}y^3 + \cdots \ldots . + a_ny^n \ldots \ldots \ldots 1)$$

In which each term is of degree n, is called as a homogeneous function of degree n in x, y.

From 1), we have therefore,

$$f(x,y) = x^n\left[a_0 + a_1\left(\frac{y}{x}\right) + a_2\left(\frac{y}{x}\right)^2 + a_3\left(\frac{y}{x}\right)^3 + \cdots \ldots + a_n\left(\frac{y}{x}\right)^n\right]$$

$$\therefore \boxed{f(x,y) = x^n\phi\left(\frac{y}{x}\right)} \ldots \ldots \ldots 2)$$

Thus, every homogeneous function of degree n in x, y can be written as 2)

A homogeneous function of degree n in x, y can also be written as $\boxed{f(x,y) = y^n\psi\left(\frac{x}{y}\right)}$

❖ **Euler's Theorem on Homogeneous Functions:**

Statement: For a homogeneous function $u(x,y)$ of degree n in (x,y),

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = nu$$

➢ **Corollary 1) (VIMP)**

If u is a homogeneous function of degree n in x, y then $x^2\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x \partial y} + y^2\frac{\partial^2 u}{\partial y^2} = n(n-1)u$

- **Corollary 2)**

  If $z = f(u)$ is a homogeneous function of degree n in x, y then $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = n\frac{f(u)}{f'(u)}$

  Note: This is used when log or Inverse trigon function come in the question.

- **Corollary 3)**

  If $z = f(u)$ is a homogeneous function of degree n in x, y then

$$x^2\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x \partial y} + y^2\frac{\partial^2 u}{\partial y^2} = g(u)[g'(u) - 1] \text{ Where } g(u) = n\frac{f(u)}{f'(u)}$$

- **Examples:**

1. If $u = \frac{x^3y^3}{x^3+y^3}$, show that $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = 3u$

   Sol: Given that $u = \frac{x^3y^3}{x^3+y^3}$

$$\therefore u = \frac{x^6\left(\frac{y^3}{x^3}\right)}{x^3\left(\frac{x^3}{x^3} + \frac{y^3}{x^3}\right)}$$

$$\therefore u = x^{6-3}\frac{\left(\frac{y^3}{x^3}\right)}{\left(1 + \frac{y^3}{x^3}\right)}$$

$$\therefore u = x^3\frac{\left(\frac{y}{x}\right)^3}{\left(1 + \left(\frac{y}{x}\right)^3\right)} = x^n\phi\left(\frac{y}{x}\right)$$

   Here u is homogeneous function of degree $n = 3$

   By Euler's Theorem.

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = nu$$

$$\boxed{\therefore x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = 3u}$$

   Hence the proof.

2. If $u = \sin^{-1}\left(\frac{x+y}{\sqrt{x}+\sqrt{y}}\right)$, prove that

   a) $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = \frac{1}{2}tanu$ b) $x^2\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x \partial y} + y^2\frac{\partial^2 u}{\partial y^2} = \frac{-sinu.cos2u}{4cos^3u}$

   Sol: Given $u = \sin^{-1}\left(\frac{x+y}{\sqrt{x}+\sqrt{y}}\right)$

   Here u is not a homogeneous function. We however write

$$\therefore sinu = \frac{x+y}{\sqrt{x}+\sqrt{y}}$$

Put $z = sinu = f(u)$

$$\therefore z = \frac{x+y}{\sqrt{x}+\sqrt{y}}$$

$$\therefore z = \frac{x\left(\frac{x}{x}+\frac{y}{x}\right)}{\sqrt{x}\left(\frac{\sqrt{x}}{\sqrt{x}}+\frac{\sqrt{y}}{\sqrt{x}}\right)}$$

$$\therefore z = \frac{x\left(1+\frac{y}{x}\right)}{x^{\frac{1}{2}}\left(1+\frac{\sqrt{y}}{\sqrt{x}}\right)} \qquad \left\{\sqrt{x} = x^{\frac{1}{2}}\right\}$$

$$\therefore z = x^{1-\frac{1}{2}}\frac{\left(1+\frac{y}{x}\right)}{\left(1+\sqrt{\frac{y}{x}}\right)} = x^{\frac{1}{2}}\frac{\left(1+\frac{y}{x}\right)}{\left(1+\sqrt{\frac{y}{x}}\right)} = x^n\phi\left(\frac{y}{x}\right)$$

Here z is a homogeneous function of degree $n = \frac{1}{2}$

(a) Then by Euler's Theorem.

$$x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y} = n\frac{f(u)}{f'(u)}$$

Here $f(u) = sinu \Rightarrow f'(u) = cosu$

$$\therefore x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y} = \frac{1}{2}\frac{sinu}{cosu}$$

Using formulae, $tanu = \frac{sinu}{cosu}$

$$\boxed{\therefore x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y} = \frac{1}{2}tanu}$$

(b) Then by Euler's Theorem.

$$x^2\frac{\partial^2 u}{\partial x^2}+2xy\frac{\partial^2 u}{\partial x\partial y}+y^2\frac{\partial^2 u}{\partial y^2} = g(u)[g'(u)-1] \text{ Where } g(u) = n\frac{f(u)}{f'(u)}$$

Here $g(u) = \frac{1}{2}tanu \Rightarrow g'(u) = \frac{1}{2}sec^2u$

$$\therefore x^2\frac{\partial^2 u}{\partial x^2}+2xy\frac{\partial^2 u}{\partial x\partial y}+y^2\frac{\partial^2 u}{\partial y^2} = \frac{1}{2}tanu\left[\frac{1}{2}sec^2u-1\right]$$

Using formulae, $tanu = \frac{sinu}{cosu}, secu = \frac{1}{cosu}$

$$\therefore x^2\frac{\partial^2 u}{\partial x^2}+2xy\frac{\partial^2 u}{\partial x\partial y}+y^2\frac{\partial^2 u}{\partial y^2} = \frac{1}{2}\frac{sinu}{cosu}\left[\frac{1}{2}\frac{1}{cos^2u}-1\right]$$

$$\therefore x^2\frac{\partial^2 u}{\partial x^2}+2xy\frac{\partial^2 u}{\partial x\partial y}+y^2\frac{\partial^2 u}{\partial y^2} = \frac{1}{2}\frac{sinu}{cosu}\left[\frac{1-2cos^2u}{2cos^2u}\right]$$

We know that, $1-2cos^2u = -cos2u$

$$\therefore x^2\frac{\partial^2 u}{\partial x^2}+2xy\frac{\partial^2 u}{\partial x\partial y}+y^2\frac{\partial^2 u}{\partial y^2} = \frac{1}{2}\frac{sinu}{cosu}\left[\frac{-cos2u}{2cos^2u}\right]$$

$$\therefore x^2\frac{\partial^2 u}{\partial x^2}+2xy\frac{\partial^2 u}{\partial x\partial y}+y^2\frac{\partial^2 u}{\partial y^2}=\frac{-sinu.cos2u}{4cos^3u}$$

Hence the proof.

3. If $u=\tan^{-1}\left(\frac{x^3+y^3}{x-y}\right)$, prove that

a) $x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}=sin2u$

b) $x^2\frac{\partial^2 u}{\partial x^2}+2xy\frac{\partial^2 u}{\partial x\partial y}+y^2\frac{\partial^2 u}{\partial y^2}=2cos3u.sinu$

Sol: Given $u=\tan^{-1}\left(\frac{x^3+y^3}{x-y}\right)$

Here u is not a homogeneous function. We however write

$$tanu=\frac{x^3+y^3}{x-y}$$

Put $z=tanu=f(u)$

$$\therefore z=\frac{x^3+y^3}{x-y}$$

$$\therefore z=\frac{x^3\left(\frac{x^3}{x^3}+\frac{y^3}{x^3}\right)}{x\left(\frac{x}{x}-\frac{y}{x}\right)}$$

$$\therefore z=x^{3-1}\frac{\left(1+\left(\frac{y}{x}\right)^3\right)}{\left(1-\frac{y}{x}\right)}=x^2\frac{\left(1+\left(\frac{y}{x}\right)^3\right)}{\left(1-\frac{y}{x}\right)}=x^n\phi\left(\frac{y}{x}\right)$$

Here z is a homogeneous function of degree $n=2$

(a) Then by Euler's Theorem.

$$x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}=n\frac{f(u)}{f'(u)}$$

Here $f(u)=tanu \Rightarrow f'(u)=sec^2u$

$$\therefore x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}=2\frac{tanu}{sec^2u}$$

Using formulae, $tanu=\frac{sinu}{cosu}, secu=\frac{1}{cosu}$

$$\therefore x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}=2\frac{\frac{sinu}{\cancel{cosu}}}{\frac{1}{cos^2u}}$$

$$\therefore x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}=2sinu.cosu$$

We know that, $2\mathrm{sinA.cosA}=\mathrm{sin2A}$

$$\therefore x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}=sin2u$$

(b) Then by Euler's Theorem.

$$x^2\frac{\partial^2 u}{\partial x^2}+2xy\frac{\partial^2 u}{\partial x\partial y}+y^2\frac{\partial^2 u}{\partial y^2}=g(u)[g'(u)-1] \text{ Where } g(u)=n\frac{f(u)}{f'(u)}$$

Here $g(u)=sin2u \Rightarrow g'(u)=cos2u.2$

$$\therefore x^2\frac{\partial^2 u}{\partial x^2}+2xy\frac{\partial^2 u}{\partial x\partial y}+y^2\frac{\partial^2 u}{\partial y^2}=sin2u\,[2.cos2u-1]$$

$$\therefore x^2\frac{\partial^2 u}{\partial x^2}+2xy\frac{\partial^2 u}{\partial x\partial y}+y^2\frac{\partial^2 u}{\partial y^2}=2.sin2u.cos2u-sin2u$$

We know that, $2\sin A.\cos A=\sin 2A \quad , A=2u$

$$\therefore x^2\frac{\partial^2 u}{\partial x^2}+2xy\frac{\partial^2 u}{\partial x\partial y}+y^2\frac{\partial^2 u}{\partial y^2}=sin4u-sin2u$$

Using formula, $sinC-sinD=2cos\left(\frac{C+D}{2}\right).sin\left(\frac{C-D}{2}\right), C=4u, D=2u$

$$\therefore x^2\frac{\partial^2 u}{\partial x^2}+2xy\frac{\partial^2 u}{\partial x\partial y}+y^2\frac{\partial^2 u}{\partial y^2}=2cos\left(\frac{4u+2u}{2}\right).sin\left(\frac{4u-2u}{2}\right)=2cos\left(\frac{6u}{2}\right)sin\left(\frac{2u}{2}\right)$$

$$\boxed{\therefore x^2\frac{\partial^2 u}{\partial x^2}+2xy\frac{\partial^2 u}{\partial x\partial y}+y^2\frac{\partial^2 u}{\partial y^2}=2cos3u.sinu}$$

Hence the proof.

4. If $u=\sin^{-1}\left(\frac{x}{y}\right)+\tan^{-1}\left(\frac{y}{x}\right)$, prove that $x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}=0$

Sol: Given $u=\sin^{-1}\left(\frac{x}{y}\right)+\tan^{-1}\left(\frac{y}{x}\right)$

Here u is not a homogeneous function. We however write

$=\sin^{-1}\left(\frac{x}{y}\right)$ $\quad$ $n=\tan^{-1}\left(\frac{y}{x}\right)$

$\therefore u=m+n\ldots\ldots 1) \Rightarrow \frac{\partial u}{\partial x}=\frac{\partial m}{\partial x}+\frac{\partial n}{\partial x}$ & $\frac{\partial u}{\partial y}=\frac{\partial m}{\partial y}+\frac{\partial n}{\partial y}$

$m=\sin^{-1}\left(\frac{x}{y}\right)$

$\therefore sinm=\frac{x}{y}$

Put $z_1=sinm=f(m)$

$\therefore z_1=y^0.\frac{x}{y}=y^n\psi\left(\frac{x}{y}\right)$

Here

z_1 is homogeneous function of degree $n=0$

Then by Euler's Theorem

$x\frac{\partial m}{\partial x}+y\frac{\partial m}{\partial y}=n\frac{f(m)}{f'(m)}$

Here, $f(m)=sinm \Rightarrow f'(m)=cosm$

$\therefore x\frac{\partial m}{\partial x}+y\frac{\partial m}{\partial y}=0.\frac{sinm}{cosm}$

$\therefore x\frac{\partial m}{\partial x}+y\frac{\partial m}{\partial y}=0\ldots\ldots 2)$

$n=\tan^{-1}\left(\frac{y}{x}\right)$

$\therefore tann=\frac{y}{x}$

$\therefore z_2=tann=f(n)$

$\therefore z_2=x^0.\frac{y}{x}=x^n\phi\left(\frac{y}{x}\right)$

z_2 is homogeneous function of degree $n=0$

Then by Euler's Theorem

$x\frac{\partial n}{\partial x}+y\frac{\partial n}{\partial y}=n\frac{f(n)}{f'(n)}$

$f(n)=tann \Rightarrow f'(n)=sec^2n$

$\therefore x\frac{\partial n}{\partial x}+y\frac{\partial n}{\partial y}=0.\frac{tann}{sec^2n}$

$\therefore x\frac{\partial n}{\partial x}+y\frac{\partial n}{\partial y}=0\ldots\ldots..3)$

Adding equation 2) & equation 3)

$$\therefore x\frac{\partial m}{\partial x}+y\frac{\partial m}{\partial y}+x\frac{\partial n}{\partial x}+y\frac{\partial n}{\partial y}=0+0$$

$$\therefore x\left(\frac{\partial m}{\partial x}+\frac{\partial n}{\partial x}\right)+y\left(\frac{\partial m}{\partial y}+\frac{\partial n}{\partial y}\right)=0$$

From 1)

$$x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}=0$$

Hence the proof.

5. **If $u=\log_e\left(\frac{x^4+y^4}{x+y}\right)$, show that $x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}=3$**

Sol: Given $u=\log_e\left(\frac{x^4+y^4}{x+y}\right)$

Here u is not a homogeneous function. We however write

$$e^u=\frac{x^4+y^4}{x+y}$$

Put $z=e^u=f(u)$

$$\therefore z=\frac{x^4+y^4}{x+y}$$

$$\therefore z=\frac{x^4\left(\frac{x^4}{x^4}+\frac{y^4}{x^4}\right)}{x\left(\frac{x}{x}+\frac{y}{x}\right)}$$

$$\therefore z=x^{4-1}\frac{\left(1+\left(\frac{y}{x}\right)^4\right)}{\left(1+\frac{y}{x}\right)}$$

$$\therefore z=x^3\frac{\left(1+\left(\frac{y}{x}\right)^4\right)}{\left(1+\frac{y}{x}\right)}=x^n\phi\left(\frac{y}{x}\right)$$

Here z is a homogeneous function of degree $n=3$

Then by Euler's theorem.

$$x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}=n\frac{f(u)}{f'(u)}$$

Here $f(u)=e^u \Rightarrow f'(u)=e^u$

$$\therefore x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}=3\frac{e^u}{e^u}$$

$$\therefore x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}=3$$

Hence the proof.

- **HOMEWORK:**

1. If $u = \cos^{-1}\left(\frac{x+y}{\sqrt{x}+\sqrt{y}}\right)$, show that $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} + \frac{1}{2}cotu = 0$
2. If $u = \sin^{-1}\left[\frac{x+2y+3z}{\sqrt{x^8+y^8+z^8}}\right]$, show that $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} + z\frac{\partial u}{\partial z} + 3tanu = 0$
3. If $u = \log_e\left(\frac{x^3+y^3}{x^2+y^2}\right)$, show that $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = 1$
4. If $u = \sec^{-1}\left(\frac{x^3-y^3}{x+y}\right)$, show that $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = 2cotu$, then evaluate $x^2\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x\partial y} + y^2\frac{\partial^2 u}{\partial y^2}$