UNIT 2

MATRICES (20 MARKS)

Topic Content: Inverse of a matrix by Gauss-Jordan method; **Rank of a matrix**; Normal form of a matrix; Consistency of non- homogeneous and homogeneous system of linear equations; Eigen values and Eigen vectors; Properties of Eigen values and Eigen vectors (without proofs); Cayley-Hamilton‘s theorem (without proof) and its applications.

MR SUDHIR S DESAI

Course Outcome: After completion of this course, students will be able to

CO2: Implement matrix concept to solve real life problems.

❖ Rank of Matrix:

The number r is called the rank of matrix A if,

1) There exist at-least one minor of order r which is non zero
2) Every minor of order $r+1$ of matrix A is zero.

❖ Note:

1) The rank of matrix A is denoted by $\rho(A)$
2) Rank of null matrix is zero.
3) If $A=\left[a_{ij}\right]_{m\times n}$ then $\rho(A)\le min(m,n)$
4) $\rho(I_n)=n$

➢ Methods of finding rank of matrix.

1) Minor Method.
2) Normal Form Method.
3) Echelon Form Method.

1) Minor (Determinant) Method:

1. Find the rank of the matrix $A=\begin{bmatrix}1 & 2 & 3\\ 4 & 5 & 6\\ 7 & 8 & 9\end{bmatrix}$ [S-23] 2M

Sol: Given $A=\begin{bmatrix}1 & 2 & 3\\ 4 & 5 & 6\\ 7 & 8 & 9\end{bmatrix}$

$$\therefore \begin{vmatrix}1 & 2 & 3\\ 4 & 5 & 6\\ 7 & 8 & 9\end{vmatrix}=(1)\begin{vmatrix}5 & 6\\ 8 & 9\end{vmatrix}-(2)\begin{vmatrix}4 & 6\\ 7 & 9\end{vmatrix}+(3)\begin{vmatrix}4 & 5\\ 7 & 8\end{vmatrix}$$

$$\therefore \begin{vmatrix}1 & 2 & 3\\ 4 & 5 & 6\\ 7 & 8 & 9\end{vmatrix}=1[45-48]-2[36-42]+3[32-35]$$

$$\therefore \begin{vmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{vmatrix} = 1[-3] - 2[-6] + 3[-3]$$

$$\therefore \begin{vmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{vmatrix} = -3 + 12 - 9$$

$$\therefore \begin{vmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{vmatrix} = 0$$

$$\therefore \rho(A) \neq 3$$

Consider any minor of A of order 2.

$$\begin{vmatrix} 4 & 5 \\ 7 & 8 \end{vmatrix} = 32 - 35 = -3 \neq 0$$

$$\boxed{\therefore \boldsymbol{\rho(A) = 2}}$$

2. **Find the rank of the matrix $A = \begin{bmatrix} \mathbf{1} & \mathbf{2} & \mathbf{3} \\ \mathbf{2} & \mathbf{4} & \mathbf{7} \\ \mathbf{3} & \mathbf{6} & \mathbf{10} \end{bmatrix}$ [W-23] 2M**

Sol: Given $A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 7 \\ 3 & 6 & 10 \end{bmatrix}$

$$\therefore \begin{vmatrix} 1 & 2 & 3 \\ 2 & 4 & 7 \\ 3 & 6 & 10 \end{vmatrix} = (1)\begin{vmatrix} 4 & 7 \\ 6 & 10 \end{vmatrix} - (2)\begin{vmatrix} 2 & 7 \\ 3 & 10 \end{vmatrix} + (3)\begin{vmatrix} 2 & 4 \\ 3 & 6 \end{vmatrix}$$

$$\therefore \begin{vmatrix} 1 & 2 & 3 \\ 2 & 4 & 7 \\ 3 & 6 & 10 \end{vmatrix} = 1.[40 - 42] - 2.[20 - 21] + 3.[12 - 12]$$

$$\therefore \begin{vmatrix} 1 & 2 & 3 \\ 2 & 4 & 7 \\ 3 & 6 & 10 \end{vmatrix} = 1.[-2] - 2.[-1] + 3.[0]$$

$$\therefore \begin{vmatrix} 1 & 2 & 3 \\ 2 & 4 & 7 \\ 3 & 6 & 10 \end{vmatrix} = -2 + 2 + 0 = 0$$

$$\therefore \rho(A) \neq 3$$

Consider any minor of A of order 2.

$$\begin{vmatrix} 4 & 7 \\ 6 & 10 \end{vmatrix} = 40 - 42 = -2 \neq 0$$

$$\boxed{\therefore \boldsymbol{\rho(A) = 2}}$$

Homework:

1. **Find the rank of the matrix $A = \begin{bmatrix} \mathbf{2} & \mathbf{4} & \mathbf{1} \\ \mathbf{3} & \mathbf{6} & \mathbf{2} \\ \mathbf{4} & \mathbf{8} & \mathbf{3} \end{bmatrix}$**

UNIT 2

MATRICES (20 MARKS)

Topic Content: Inverse of a matrix by Gauss-Jordan method; **Rank of a matrix (Echelon Form)**; Normal form of a matrix; Consistency of non- homogeneous and homogeneous system of linear equations; Eigen values and Eigen vectors; Properties of Eigen values and Eigen vectors (without proofs); Cayley-Hamilton's theorem (without proof) and its applications.

MR SUDHIR S DESAI

Course Outcome: After completion of this course, students will be able to

CO2: Implement matrix concept to solve real life problems.

❖ Echelon Form:

Using the row elementary operations, we can transform a given non-zero matrix to a simplified form called a **Row-echelon form**. In a row-echelon form, we may have rows all of whose entries are zero. Such rows are called **zero rows**. A non-zero row is one in which at least one of the entries is not zero. For instance, in the matrix, $\begin{bmatrix} 6 & 0 & -1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$

R_1 and R_2 are non-zero rows and R_3 is a zero row.

Step 1

Inspect the first row. If the first row is a zero row, then the row is interchanged with a non-zero row below the first row. If a_{11} is not equal to 0, then go to step 2. Otherwise, interchange the first row R_1 with any other row below the first row which has a non-zero element in the first column; if no row below the first row has non-zero entry in the first column, then consider a_{12} . If a_{12} is not equal to 0, then go to step 2. Otherwise, interchange the first row R_1 with any other row below the first row which has a non-zero element in the second column; if no row below the first row has non-zero entry in the second column, then consider a_{13}. Proceed in the same way till we get a non-zero entry in the first row. This is called **pivoting** and the first non-zero element in the first row is called the **pivot** of the first row.

Step 2

Use the first row and elementary row operations to transform all elements under the pivot to become zeroes.

Step 3

Consider the next row as first row and perform steps 1 and 2 with the rows below this row only.

Repeat the step until all rows are exhausted.

❖ Rank of Matrix:

The rank of a matrix in row echelon form is the number of non-zero rows in it.

❖ **The elementary row operations include:**

- Swapping two rows.
- Multiplying a row by a non-zero scalar.
- Adding or subtracting the multiple of one row from another row.

❖ **Examples:**

1. **Reduce the given matrix to echelon form $A = \begin{bmatrix} 1 & 1 & 2 \\ 1 & 2 & 2 \\ 2 & 2 & 3 \end{bmatrix}$ and find its rank. [S-24 4M]**

Sol: Given $A = \begin{bmatrix} 1 & 1 & 2 \\ 1 & 2 & 2 \\ 2 & 2 & 3 \end{bmatrix}$

$\boldsymbol{R_2 \to R_2 - R_1}$

$\boldsymbol{R_2 - R_1}$

$\mathbf{1 - 1 = 0}$

$\mathbf{2 - 1 = 1}$

$\mathbf{2 - 2 = 0}$

$\therefore A = \begin{bmatrix} 1 & 1 & 2 \\ 0 & 1 & 0 \\ 2 & 2 & 3 \end{bmatrix}$

$\boldsymbol{R_3 \to R_3 - 2 \times R_1}$

$\boldsymbol{R_3 - 2 \times R_1}$

$\mathbf{2 - 2 \times 1 = 0}$

$\mathbf{2 - 2 \times 1 = 0}$

$\mathbf{3 - 2 \times 2 = -1}$

$\therefore A = \begin{bmatrix} 1 & 1 & 2 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$

This is echelon form.

Number of non-zero rows are 3

The rank of matrix A is $\boxed{\rho(A) = 3}$

2. **Reduce the given matrix to ECHELON form $A = \begin{bmatrix} 1 & 1 & 2 \\ 1 & 2 & 3 \\ 3 & 4 & 5 \end{bmatrix}$ and find its rank. [W-23 4M]**

Sol: Given $A = \begin{bmatrix} 1 & 1 & 2 \\ 1 & 2 & 3 \\ 3 & 4 & 5 \end{bmatrix}$

$\boldsymbol{R_2 \to R_2 - R_1}$

$\boldsymbol{R_2 - R_1}$

$\mathbf{1 - 1 = 0}$

$\mathbf{2 - 1 = 1}$

$\mathbf{3 - 2 = 1}$

$\therefore A = \begin{bmatrix} 1 & 1 & 2 \\ 0 & 1 & 1 \\ 3 & 4 & 5 \end{bmatrix}$

$R_3 \to R_3 - 3 \times R_1$

$R_3 - 3 \times R_1$

$3 - 3 \times 1 = 0$

$4 - 3 \times 1 = 1$

$5 - 3 \times 2 = -1$

$\therefore A = \begin{bmatrix} 1 & 1 & 2 \\ 0 & 1 & 1 \\ 0 & 1 & -1 \end{bmatrix}$

$R_3 \to R_3 - R_2$

$R_3 - R_2$

$0 - 0 = 0$

$1 - 1 = 0$

$-1 - 1 = -2$

$\therefore A = \begin{bmatrix} 1 & 1 & 2 \\ 0 & 1 & 1 \\ 0 & 0 & -2 \end{bmatrix}$

This is echelon form.

Number of non-zero rows are 3

The rank of matrix A is $\boxed{\rho(A) = 3}$

3. Reduce the given matrix to ECHELON form $A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 1 & 4 \\ 3 & 0 & 5 \end{bmatrix}$ and find its rank

Sol: Given $A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 1 & 4 \\ 3 & 0 & 5 \end{bmatrix}$

$R_2 \to R_2 - 2 \times R_1$

$R_2 - 2 \times R_1$

$2 - 2 \times 1 = 0$

$1 - 2 \times 2 = -3$

$4 - 2 \times 3 = -2$

$\therefore A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & -3 & -2 \\ 3 & 0 & 5 \end{bmatrix}$

$R_3 \to R_3 - 3 \times R_1$

$R_3 - 3 \times R_1$

$3 - 3 \times 1 = 0$

$0 - 3 \times 2 = -6$

$5 - 3 \times 3 = -4$

$$\therefore A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & -3 & -2 \\ 0 & -6 & -4 \end{bmatrix}$$

$R_3 \rightarrow R_3 - 2 \times R_2$

$R_3 - 2 \times R_2$

$0 - 2 \times 0 = 0$

$-6 - 2 \times -3 = 0$

$-4 - 2 \times -2 = 0$

$$\therefore A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & -3 & -2 \\ 0 & 0 & 0 \end{bmatrix}$$

This is echelon form.

Number of non-zero rows are 2

The rank of matrix A is $\boxed{\rho(A) = 2}$

- **Homework:**

Reduce the given matrix to ECHELON form and find its rank

1. $A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$
2. $A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 4 \\ 3 & 4 & 5 \end{bmatrix}$
3. $A = \begin{bmatrix} 1 & 3 & 2 \\ 2 & 6 & 4 \\ -1 & -3 & -2 \end{bmatrix}$
4. $A = \begin{bmatrix} 4 & 1 & 2 \\ 1 & 3 & 4 \\ 2 & 4 & 3 \end{bmatrix}$
5. $A = \begin{bmatrix} 2 & -3 & 5 \\ 6 & -9 & 15 \\ 8 & -12 & 20 \end{bmatrix}$

UNIT 2

MATRICES (20 MARKS)

Topic Content: Inverse of a matrix by Gauss-Jordan method; **Rank of a matrix (Normal Form)**; Normal form of a matrix; Consistency of non- homogeneous and homogeneous system of linear equations; Eigen values and Eigen vectors; Properties of Eigen values and Eigen vectors (without proofs); Cayley-Hamilton's theorem (without proof) and its applications.

MR SUDHIR S DESAI

Course Outcome: After completion of this course, students will be able to

CO2: Implement matrix concept to solve real life problems.

❖ Normal Form:

Any non-zero matrix A can be reduced to following four forms,

$I_r, [I_r \quad 0], \begin{bmatrix} I_r \\ 0 \end{bmatrix}, \begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix}$ Where I_r is the identity matrix of order r. These forms are called Normal Form.

Note: Rank of matrix in normal form $=\ r$, since$|I_r| = 1 \neq 0$.

Procedure to write matrix in Normal Form:

Step (1): By using elementary transformation make $a_{11} = 1$

Step (2): Using row/ column transformation make 1st column and 1st row zero except a_{11}.

Step (3): Use the same procedure on a_{22} and diagonal elements still we get normal form.

❖ The elementary row operations include:

- **Swapping two rows.**
- **Multiplying a row by a non-zero scalar.**
- **Adding or subtracting the multiple of one row from another row.**

❖ Examples:

4. Reduce the following matrix to normal form and hence find its rank.

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 4 \\ 3 & 4 & 5 \end{bmatrix} \text{ [S-23 4M]}$$

Sol: Given $A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 4 \\ 3 & 4 & 5 \end{bmatrix}$

$\boldsymbol{R_2 \to R_2 - 2 \times R_1} \qquad \boldsymbol{R_3 \to R_3 - 3 \times R_1}$

$$R_2 - 2 \times R_1 \qquad R_3 - 3 \times R_1$$

$$2 - 2 \times 1 = 0 \qquad 3 - 3 \times 1 = 0$$

$$3 - 2 \times 2 = -1 \qquad 4 - 3 \times 2 = -2$$

$$4 - 2 \times 3 = -2 \qquad 5 - 3 \times 3 = -4$$

$$\therefore A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & -1 & -2 \\ 0 & -2 & -4 \end{bmatrix}$$

$$C_2 \to C_2 - 2 \times C_1 \qquad C_3 \to C_3 - 3 \times C_1$$

$$C_2 - 2 \times C_1 \qquad C_3 - 3 \times C_1$$

$$2 - 2 \times 1 = 0 \qquad 3 - 3 \times 1 = 0$$

$$-1 - 2 \times 0 = -1 \qquad -2 - 3 \times 0 = -2$$

$$-2 - 2 \times 0 = -2 \qquad -4 - 3 \times 0 = -4$$

$$\therefore A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & -2 \\ 0 & -2 & -4 \end{bmatrix}$$

$$R_2 \to -1 \times R_2, R_3 \to -\frac{1}{2} \times R_3$$

$$\therefore A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 2 \\ 0 & 1 & 2 \end{bmatrix}$$

$$R_3 \to R_3 - R_2$$

$$R_3 - R_2$$

$$0 - 0 = 0$$

$$1 - 1 = 0$$

$$2 - 2 = 0$$

$$\therefore A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix}$$

$$C_3 \to C_3 - 2 \times C_2$$

$$C_3 - 2 \times C_2$$

$$0 - 2 \times 0 = 0$$

$$2 - 2 \times 1 = 0$$

$$0 - 2 \times 0 = 0$$

$$\therefore A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} I_2 & 0 \\ 0 & 0 \end{bmatrix}$$

This is normal form.

Number of non-zero rows are 2

The rank of matrix A is $\boxed{\rho(A) = 2}$

5. Reduce the given matrix to normal form $A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 1 & 4 \\ 3 & 0 & 5 \end{bmatrix}$ and find its rank

Sol: Given $A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 1 & 4 \\ 3 & 0 & 5 \end{bmatrix}$

$R_2 \to R_2 - 2 \times R_1$ $\quad$ $R_3 \to R_3 - 3 \times R_1$

$R_2 - 2 \times R_1$	$R_3 - 3 \times R_1$
$2 - 2 \times 1 = 0$	$3 - 3 \times 1 = 0$
$1 - 2 \times 2 = -3$	$0 - 3 \times 2 = -6$
$4 - 2 \times 3 = -2$	$5 - 3 \times 3 = -4$

$\therefore A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & -3 & -2 \\ 0 & -6 & -4 \end{bmatrix}$

$C_2 \to C_2 - 2 \times C_1$ $\quad$ $C_3 \to C_3 - 3 \times C_1$

$C_2 - 2 \times C_1$	$C_3 - 3 \times C_1$
$2 - 2 \times 1 = 0$	$3 - 3 \times 1 = 0$
$-3 - 2 \times 0 = -3$	$-2 - 3 \times 0 = -2$
$-6 - 2 \times 0 = -6$	$-4 - 3 \times 0 = -4$

$\therefore A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -3 & -2 \\ 0 & -6 & -4 \end{bmatrix}$

$R_2 \to -1 \times R_2, R_3 \to -\frac{1}{2} \times R_3$

$\therefore A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 3 & 2 \\ 0 & 3 & 2 \end{bmatrix}$

$C_2 \to \frac{1}{3} \times C_2$

$\therefore A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 2 \\ 0 & 1 & 2 \end{bmatrix}$

$R_3 \to R_3 - R_2$

$R_3 - R_2$

$0 - 0 = 0$

$1 - 1 = 0$

$2 - 2 = 0$

$\therefore A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix}$

$C_3 \to C_3 - 2 \times C_2$

$C_3 - 2 \times C_2$

$0 - 2 \times 0 = 0$

$2 - 2 \times 1 = 0$

$0 - 2 \times 0 = 0$

$$\therefore A = \begin{bmatrix} \mathbf{1} & \mathbf{0} & 0 \\ \mathbf{0} & \mathbf{1} & 0 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} I_2 & 0 \\ 0 & 0 \end{bmatrix}$$

This is normal form.

Number of non-zero rows are 2

The rank of matrix A is $\boxed{\rho(A) = 2}$

6. **Reduce the given matrix to normal form** $A = \begin{bmatrix} \mathbf{2} & \mathbf{-3} & \mathbf{5} \\ \mathbf{6} & \mathbf{-9} & \mathbf{15} \\ \mathbf{8} & \mathbf{-12} & \mathbf{20} \end{bmatrix}$ **and find its rank**

Sol: Given $A = \begin{bmatrix} 2 & -3 & 5 \\ 6 & -9 & 15 \\ 8 & -12 & 20 \end{bmatrix}$

$\boldsymbol{C_1 \to C_3 - 2 \times C_1}$

$\boldsymbol{C_3 - 2 \times C_1}$

$5 - 2 \times 2 = 1$

$15 - 2 \times 6 = 3$

$20 - 2 \times 8 = 4$

$$\therefore A = \begin{bmatrix} 1 & -3 & 5 \\ 3 & -9 & 15 \\ 4 & -12 & 20 \end{bmatrix}$$

$\boldsymbol{R_2 \to R_2 - 3 \times R_1}$ $\quad$ $\boldsymbol{R_3 \to R_3 - 4 \times R_1}$

$\boldsymbol{R_2 - 3 \times R_1}$ $\quad$ $\boldsymbol{R_3 - 4 \times R_1}$

$3 - 3 \times 1 = 0$ $\quad$ $4 - 4 \times 1 = 0$

$-9 - 3 \times -3 = 0$ $\quad$ $-12 - 4 \times -3 = 0$

$15 - 3 \times 5 = 0$ $\quad$ $20 - 4 \times 5 = 0$

$$\therefore A = \begin{bmatrix} 1 & -3 & 5 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$\boldsymbol{C_2 \to C_2 + 3 \times C_1}$ $\quad$ $\boldsymbol{C_3 \to C_3 - 5 \times C_1}$

$\boldsymbol{C_2 + 3 \times C_1}$ $\quad$ $\boldsymbol{C_3 - 5 \times C_1}$

$-3 + 3 \times 1 = 0$ $\quad$ $5 - 5 \times 1 = 0$

$0 + 3 \times 0 = 0$ $\quad$ $0 - 5 \times 0 = 0$

$0 + 3 \times 0 = 0$ $\quad$ $0 - 5 \times 0 = 0$

$$\therefore A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

This is normal form.

Number of non-zero rows are 1

The rank of matrix A is $\boxed{\rho(A) = 1}$

- **Homework:**

Reduce the given matrix to Normal form and find its rank

6. $A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$

7. $A = \begin{bmatrix} 1 & 1 & 2 \\ 1 & 2 & 3 \\ 3 & 4 & 5 \end{bmatrix}$

8. $A = \begin{bmatrix} 1 & 3 & 2 \\ 2 & 6 & 4 \\ -1 & -3 & -2 \end{bmatrix}$

9. $A = \begin{bmatrix} 4 & 1 & 2 \\ 1 & 3 & 4 \\ 2 & 4 & 3 \end{bmatrix}$

UNIT 2

MATRICES (12 MARKS)

Topic Content: Inverse of a matrix by Gauss-Jordan method; Rank of a matrix; Normal form of a matrix; Consistency of non- homogeneous and homogeneous system of linear equations; Eigen values and Eigen vectors; Properties of Eigen values and Eigen vectors (without proofs); Cayley-Hamilton's theorem (without proof) and its applications.

MR SUDHIR S DESAI

Course Outcome: After completion of this course, students will be able to

CO2: Implement matrix concept to solve real life problems.

❖ Inverse of a matrix by Gauss-Jordan method:

STEP I: **Write the given matrix in the form. $A.A^{-1} = I$, where I is the identity matrix.** $(|A| \neq 0)$

STEP II: **Use elementary row operations to transform the left side of the matrix (which is A) into the identity matrix. Apply same transformation to right on the matrix I into the A^{-1}**

❖ The elementary row operations include:

- Swapping two rows.
- Multiplying a row by a non-zero scalar.
- Adding or subtracting the multiple of one row from another row.

EXAMPLE:

1. **Solve by using Gauss Jordan method Find inverse of matrix $A = \begin{bmatrix} 3 & -3 & 4 \\ 2 & -3 & 4 \\ 0 & -1 & 1 \end{bmatrix}$ [W-23 4M]**

Sol: Write the matrix in the form. $A.A^{-1} = I$

$$\therefore \begin{bmatrix} 3 & -3 & 4 \\ 2 & -3 & 4 \\ 0 & -1 & 1 \end{bmatrix} . A^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$R_1 \to R_1 - R_2$

$$\therefore \begin{bmatrix} 1 & 0 & 0 \\ 2 & -3 & 4 \\ 0 & -1 & 1 \end{bmatrix} . A^{-1} = \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$R_2 \to R_2 - 2 * R_1$

$$\therefore \begin{bmatrix} 1 & 0 & 0 \\ 0 & -3 & 4 \\ 0 & -1 & 1 \end{bmatrix} . A^{-1} = \begin{bmatrix} 1 & -1 & 0 \\ -2 & 3 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$R_2 \leftrightarrow R_3$

$$\therefore \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 1 \\ 0 & -3 & 4 \end{bmatrix} . A^{-1} = \begin{bmatrix} 1 & -1 & 0 \\ 0 & 0 & 1 \\ -2 & 3 & 0 \end{bmatrix}$$

$R_2 \to -R_2$

$$\therefore \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & -3 & 4 \end{bmatrix} . A^{-1} = \begin{bmatrix} 1 & -1 & 0 \\ 0 & 0 & -1 \\ -2 & 3 & 0 \end{bmatrix}$$

$R_3 \rightarrow R_3 + 3 * R_2$

$$\therefore \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix} . A^{-1} = \begin{bmatrix} 1 & -1 & 0 \\ 0 & 0 & -1 \\ -2 & 3 & -3 \end{bmatrix}$$

$R_2 \rightarrow R_2 + R_3$

$$\therefore \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} . A^{-1} = \begin{bmatrix} 1 & -1 & 0 \\ -2 & 3 & -4 \\ -2 & 3 & -3 \end{bmatrix}$$

$$\therefore I . A^{-1} = \begin{bmatrix} 1 & -1 & 0 \\ -2 & 3 & -4 \\ -2 & 3 & -3 \end{bmatrix}$$

$$\boxed{\therefore \boldsymbol{A^{-1}} = \begin{bmatrix} \mathbf{1} & \mathbf{-1} & \mathbf{0} \\ \mathbf{-2} & \mathbf{3} & \mathbf{-4} \\ \mathbf{-2} & \mathbf{3} & \mathbf{-3} \end{bmatrix}}$$

2. Use Gauss Jordan method to Find inverse of matrix $A = \begin{bmatrix} \mathbf{2} & \mathbf{1} & \mathbf{-1} \\ \mathbf{0} & \mathbf{2} & \mathbf{-1} \\ \mathbf{5} & \mathbf{2} & \mathbf{-3} \end{bmatrix}$ (Dec-2018)

Sol: Write the matrix in the form. $A . A^{-1} = I$

$$\therefore \begin{bmatrix} 2 & 1 & -1 \\ 0 & 2 & -1 \\ 5 & 2 & -3 \end{bmatrix} . A^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$R_1 \rightarrow \frac{R_1}{2}$

$$\therefore \begin{bmatrix} 1 & \frac{1}{2} & \frac{-1}{2} \\ 0 & 2 & -1 \\ 5 & 2 & -3 \end{bmatrix} . A^{-1} = \begin{bmatrix} \frac{1}{2} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$R_3 \rightarrow R_3 - 5 * R_1$

$$\therefore \begin{bmatrix} 1 & \frac{1}{2} & \frac{-1}{2} \\ 0 & 2 & -1 \\ 0 & -\frac{1}{2} & \frac{-1}{2} \end{bmatrix} . A^{-1} = \begin{bmatrix} \frac{1}{2} & 0 & 0 \\ 0 & 1 & 0 \\ \frac{-5}{2} & 0 & 1 \end{bmatrix}$$

$R_2 \rightarrow \frac{R_2}{2}$

$$\therefore \begin{bmatrix} 1 & \frac{1}{2} & \frac{-1}{2} \\ 0 & 1 & \frac{-1}{2} \\ 0 & -\frac{1}{2} & \frac{-1}{2} \end{bmatrix} . A^{-1} = \begin{bmatrix} \frac{1}{2} & 0 & 0 \\ 0 & \frac{1}{2} & 0 \\ \frac{-5}{2} & 0 & 1 \end{bmatrix}$$

$R_1 \rightarrow R_1 + R_3$

$$\therefore \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & \frac{-1}{2} \\ 0 & -\frac{1}{2} & \frac{-1}{2} \end{bmatrix} . A^{-1} = \begin{bmatrix} -2 & 0 & 1 \\ 0 & \frac{1}{2} & 0 \\ \frac{-5}{2} & 0 & 1 \end{bmatrix}$$

$R_3 \rightarrow R_3 + \frac{1}{2} * R_2$

$$\therefore \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & \frac{-1}{2} \\ 0 & 0 & \frac{-3}{4} \end{bmatrix} . A^{-1} = \begin{bmatrix} -2 & 0 & 1 \\ 0 & \frac{1}{2} & 0 \\ \frac{-5}{2} & \frac{1}{4} & 1 \end{bmatrix}$$

$$R_3 \to \frac{-4}{3} * R_3$$

$$\therefore \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & \frac{-1}{2} \\ 0 & 0 & 1 \end{bmatrix} . A^{-1} = \begin{bmatrix} -2 & 0 & 1 \\ 0 & \frac{1}{2} & 0 \\ \frac{10}{3} & \frac{-1}{3} & \frac{-4}{3} \end{bmatrix}$$

$$R_1 \to R_1 + R_3$$

$$\therefore \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & \frac{-1}{2} \\ 0 & 0 & 1 \end{bmatrix} . A^{-1} = \begin{bmatrix} \frac{4}{3} & \frac{-1}{3} & \frac{-1}{3} \\ 0 & \frac{1}{2} & 0 \\ \frac{10}{3} & \frac{-1}{3} & \frac{-4}{3} \end{bmatrix}$$

$$R_2 \to R_2 + \frac{1}{2} * R_3$$

$$\therefore \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} . A^{-1} = \begin{bmatrix} \frac{4}{3} & \frac{-1}{3} & \frac{-1}{3} \\ \frac{5}{3} & \frac{1}{3} & \frac{-2}{3} \\ \frac{10}{3} & \frac{-1}{3} & \frac{-4}{3} \end{bmatrix}$$

$$\therefore I . A^{-1} = \begin{bmatrix} \frac{4}{3} & \frac{-1}{3} & \frac{-1}{3} \\ \frac{5}{3} & \frac{1}{3} & \frac{-2}{3} \\ \frac{10}{3} & \frac{-1}{3} & \frac{-4}{3} \end{bmatrix}$$

$$\boxed{\therefore \boldsymbol{A^{-1}} = \begin{bmatrix} \frac{4}{3} & \frac{-1}{3} & \frac{-1}{3} \\ \frac{5}{3} & \frac{1}{3} & \frac{-2}{3} \\ \frac{10}{3} & \frac{-1}{3} & \frac{-4}{3} \end{bmatrix}}$$

3. Use Gauss Jordan method to Find inverse of matrix $A = \begin{bmatrix} 2 & 0 & -1 \\ 5 & 1 & 0 \\ 0 & 1 & 3 \end{bmatrix}$

Sol: Write the matrix in the form. $A . A^{-1} = I$

$$\therefore \begin{bmatrix} 2 & 0 & -1 \\ 5 & 1 & 0 \\ 0 & 1 & 3 \end{bmatrix} . A^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$R_1 \to \frac{R_1}{2}$$

$$\therefore \begin{bmatrix} 1 & 0 & \frac{-1}{2} \\ 5 & 1 & 0 \\ 0 & 1 & 3 \end{bmatrix} . A^{-1} = \begin{bmatrix} \frac{1}{2} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$R_2 \to R_2 - 5 * R_1$

$$\therefore \begin{bmatrix} 1 & 0 & \frac{-1}{2} \\ 0 & 1 & \frac{5}{2} \\ 0 & 1 & 3 \end{bmatrix} . A^{-1} = \begin{bmatrix} \frac{1}{2} & 0 & 0 \\ \frac{-5}{2} & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$R_3 \to R_3 - R_2$

$$\therefore \begin{bmatrix} 1 & 0 & \frac{-1}{2} \\ 0 & 1 & \frac{5}{2} \\ 0 & 0 & \frac{1}{2} \end{bmatrix} . A^{-1} = \begin{bmatrix} \frac{1}{2} & 0 & 0 \\ \frac{-5}{2} & 1 & 0 \\ \frac{5}{2} & -1 & 1 \end{bmatrix}$$

$R_3 \to 2 * R_3$

$$\therefore \begin{bmatrix} 1 & 0 & \frac{-1}{2} \\ 0 & 1 & \frac{5}{2} \\ 0 & 0 & 1 \end{bmatrix} . A^{-1} = \begin{bmatrix} \frac{1}{2} & 0 & 0 \\ \frac{-5}{2} & 1 & 0 \\ 5 & -2 & 2 \end{bmatrix}$$

$R_1 \to R_1 + \frac{1}{2} * R_3$

$$\therefore \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & \frac{5}{2} \\ 0 & 0 & 1 \end{bmatrix} . A^{-1} = \begin{bmatrix} 3 & -1 & 1 \\ \frac{-5}{2} & 1 & 0 \\ 5 & -2 & 2 \end{bmatrix}$$

$R_2 \to R_2 - \frac{5}{2} * R_3$

$$\therefore \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} . A^{-1} = \begin{bmatrix} 3 & -1 & 1 \\ -15 & 6 & -5 \\ 5 & -2 & 2 \end{bmatrix}$$

$$\therefore I . A^{-1} = \begin{bmatrix} 3 & -1 & 1 \\ -15 & 6 & -5 \\ 5 & -2 & 2 \end{bmatrix}$$

$$\boxed{\therefore \boldsymbol{A^{-1}} = \begin{bmatrix} \mathbf{3} & \mathbf{-1} & \mathbf{1} \\ \mathbf{-15} & \mathbf{6} & \mathbf{-5} \\ \mathbf{5} & \mathbf{-2} & \mathbf{2} \end{bmatrix}}$$

4. Use Gauss Jordan method to Find inverse of matrix $A = \begin{bmatrix} 1 & 2 & -2 \\ 0 & -2 & 1 \\ -1 & 3 & 0 \end{bmatrix}$ [S-24, 4M]

Sol: Write the matrix in the form. $A . A^{-1} = I$

$$\therefore \begin{bmatrix} 1 & 2 & -2 \\ 0 & -2 & 1 \\ -1 & 3 & 0 \end{bmatrix} . A^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$R_3 \to R_3 + R_1$

$$\therefore \begin{bmatrix} 1 & 2 & -2 \\ 0 & -2 & 1 \\ 0 & 5 & -2 \end{bmatrix} . A^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$$

$R_2 \rightarrow 2 * R_2 + R_3$

$$\therefore \begin{bmatrix} 1 & 2 & -2 \\ 0 & 1 & 0 \\ 0 & 5 & -2 \end{bmatrix} . A^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 2 & 1 \\ 1 & 0 & 1 \end{bmatrix}$$

$R_1 \rightarrow R_1 - 2 * R_2, R_3 \rightarrow R_3 - 5 * R_2$

$$\therefore \begin{bmatrix} 1 & 0 & -2 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{bmatrix} . A^{-1} = \begin{bmatrix} -1 & -4 & -2 \\ 1 & 2 & 1 \\ -4 & -10 & -4 \end{bmatrix}$$

$R_3 \rightarrow \dfrac{R_3}{-2}$

$$\therefore \begin{bmatrix} 1 & 0 & -2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} . A^{-1} = \begin{bmatrix} -1 & -4 & -2 \\ 1 & 2 & 1 \\ 2 & 5 & 2 \end{bmatrix}$$

$R_1 \rightarrow R_1 + 2 * R_3$

$$\therefore \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} . A^{-1} = \begin{bmatrix} 3 & 6 & 2 \\ 1 & 2 & 1 \\ 2 & 5 & 2 \end{bmatrix}$$

$$\therefore I . A^{-1} = \begin{bmatrix} 3 & 6 & 2 \\ 1 & 2 & 1 \\ 2 & 5 & 2 \end{bmatrix}$$

$$\boxed{\therefore \boldsymbol{A^{-1}} = \begin{bmatrix} \mathbf{3} & \mathbf{6} & \mathbf{2} \\ \mathbf{1} & \mathbf{2} & \mathbf{1} \\ \mathbf{2} & \mathbf{5} & \mathbf{2} \end{bmatrix}}$$

Homework:

Use Gauss Jordan method to find the inverse of matrix A

1. $\begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 5 \\ 3 & 5 & 6 \end{bmatrix}$

2. $\begin{bmatrix} 1 & 3 & 0 \\ -1 & 2 & 1 \\ 0 & 0 & 2 \end{bmatrix}$

3. $\begin{bmatrix} 1 & 1 & 3 \\ 1 & 3 & -3 \\ -2 & -4 & -4 \end{bmatrix}$

4. $\begin{bmatrix} 2 & 3 & 4 \\ 4 & 3 & 1 \\ 1 & 2 & 4 \end{bmatrix}$

5. $\begin{bmatrix} 1 & 2 & -2 \\ -1 & 3 & 0 \\ 0 & -2 & 1 \end{bmatrix}$ **[S-23 4M]**

UNIT 2

MATRICES (20 MARKS)

Topic Content: Inverse of a matrix by Gauss-Jordan method; Rank of a matrix; Normal form of a matrix; **Consistency of non- homogeneous and homogeneous system of linear equations**; Eigen values and Eigen vectors; Properties of Eigen values and Eigen vectors (without proofs); Cayley-Hamilton's theorem (without proof) and its applications.

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Course Outcome: After completion of this course, students will be able to

CO2: Implement matrix concept to solve real life problems.

❖ Consistency of non- homogeneous and homogeneous system of linear equations:

Let us consider the non-homogeneous system of m linear equations in n unknowns $x_1, x_2, x_3, \dots \dots \dots, x_n$.

$$\left.\begin{array}{l} a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + \cdots + a_{1n}x_n = b_1, \\ a_{21}x_1 + a_{22}x_2 + a_{23}x_3 + \cdots + a_{2n}x_n = b_2, \\ \vdots \quad\quad \vdots \quad\quad \vdots \quad\quad\quad \vdots \quad\quad \vdots \end{array}\right\} \quad 1)$$

$$a_{m1}x_1 + a_{m2}x_2 + a_{m3}x_3 + \cdots + a_{mn}x_n = b_m$$

To check the consistency (i.e. if the system of equation has at least one solution) we find the rank of matrices.

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} & \dots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \dots & a_{2n} \\ \vdots & \vdots & \vdots & \dots & \vdots \\ a_{m1} & a_{m2} & a_{m3} & \dots & a_{mn} \end{bmatrix}$$

And

$$K = [A:B] = \left[\begin{array}{ccccc|c} a_{11} & a_{12} & a_{13} & \dots & a_{1n} & b_1 \\ a_{21} & a_{22} & a_{23} & \dots & a_{2n} & b_2 \\ \vdots & \vdots & \vdots & \dots & \vdots & \vdots \\ a_{m1} & a_{m2} & a_{m3} & \dots & a_{mn} & b_m \end{array}\right]$$

Here A is called coefficient matrix & K is called augmented matrix of equation 1)

We find the rank of matrix A & K by reducing them to triangular form by applying elementary row transformations.

➢ Conditions To Check Consistency:

Case I) If $\rho(A) \neq \rho(K)$ then the system of equation 1) is said to be inconsistent. There will be no solution.

Case II) If $\rho(A) = \rho(K)$ then the system of equation 1) is said to be consistent. There will be at least one solution.

1) $\rho(A) = \rho(K) = n$ (**Number of unknowns**) the system will have a unique solution.
2) $\rho(A) = \rho(K) < n$ (**Number of unknowns**) the system will have infinite number of solution & assign $n - r$ parameters to variables.

Examples:

1. **Determine the consistency of set of equations.** $\boldsymbol{x - 2y + z = -5, x + 5y - 7z = 2, 3x + y - 5z = 1}$.

   Sol: The given system of equation can be written as, $AX = B$

$$\therefore \begin{bmatrix} 1 & -2 & 1 \\ 1 & 5 & -7 \\ 3 & 1 & -5 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -5 \\ 2 \\ 1 \end{bmatrix}$$

$$R_2 \to R_2 - R_1, R_3 \to R_3 - 3 \times R_1$$

$\boldsymbol{R_2 - R_1}$	$\boldsymbol{R_3 - 3 \times R_1}$
$1 - 1 = 0$	$3 - 3 \times 1 = 0$
$5 - (-2) = 7$	$1 - 3 \times (-2) = 7$
$-7 - 1 = -8$	$-5 - 3 \times 1 = -8$
$2 - (-5) = 7$	$1 - 3 \times (-5) = 16$

$$\therefore \begin{bmatrix} 1 & -2 & 1 \\ 0 & 7 & -8 \\ 0 & 7 & -8 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -5 \\ 7 \\ 16 \end{bmatrix}$$

$$R_3 \to R_3 - R_2$$

$\boldsymbol{R_3 - R_2}$
$0 - 0 = 0$
$7 - 7 = 7)$
$-8 - (-8) = 0$
$16 - 7 = 9$

$$\therefore \begin{bmatrix} 1 & -2 & 1 \\ 0 & 7 & -8 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -5 \\ 7 \\ 9 \end{bmatrix}$$

$$\text{Here } \rho(A) = 2 \;\&\; \rho(K) = \left[\begin{array}{ccc|c} 1 & -2 & 1 & -5 \\ 0 & 7 & -8 & 7 \\ 0 & 0 & 0 & 9 \end{array}\right] = 3.$$

$$\therefore \rho(A) = 2 \neq \rho(K) = 3$$

System is in consistent & has no solution.

2. **Test for consistency and solve:** $\boldsymbol{5x + 3y + 7z = 4, 3x + 26y + 2z = 9, 7x + 2y + 10z = 5}$ **[S-24 4M]**

   Sol: The given system of equation can be written as, $AX = B$

$$\therefore \begin{bmatrix} 5 & 3 & 7 \\ 3 & 26 & 2 \\ 7 & 2 & 10 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 4 \\ 9 \\ 5 \end{bmatrix}$$

$$R_2 \to 5 \times R_2 - 3 \times R_1, R_3 \to 5 \times R_3 - 7 \times R_1$$

$\boldsymbol{5 \times R_2 - 3 \times R_1}$	$\boldsymbol{5 \times R_3 - 7 \times R_1}$
$5 \times 3 - 3 \times 5 = 0$	$5 \times 7 - 7 \times 5 = 0$
$5 \times 26 - 3 \times 3 = 121$	$5 \times 2 - 7 \times 3 = -11$
$5 \times 2 - 3 \times 7 = -11$	$5 \times 10 - 7 \times 7 = 1$
$5 \times 9 - 3 \times 4 = 33$	$5 \times 5 - 7 \times 4 = -3$

$$\therefore \begin{bmatrix} 5 & 3 & 7 \\ 0 & 121 & -11 \\ 0 & -11 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 4 \\ 33 \\ -3 \end{bmatrix}$$

$R_3 \to 11 \times R_3 + R_2$

$\mathbf{11 \times R_3 - R_2}$
$11 \times 0 - 0 = 0$
$11 \times -11 + 121 = 0$
$11 \times 1 + (-11) = 0$
$11 \times -3 + 33 = 0$

$$\therefore \begin{bmatrix} 5 & 3 & 7 \\ 0 & 121 & -11 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 4 \\ 33 \\ 0 \end{bmatrix} \ldots\ldots\ldots(1)$$

$$\text{Here } \rho(A) = 2 \ \&\ \rho(K) = \left[\begin{array}{ccc|c} 5 & 3 & 7 & 4 \\ 0 & 121 & -11 & 33 \\ 0 & 0 & 0 & 0 \end{array}\right] = 2.$$

$$\therefore \rho(A) = 2 = \rho(K) = 2$$

This system is consistent.

Assign $n - r = 3 - 2 = 1$ parameter.

Consider $\boxed{z = t}$

From equation 1)

$$\therefore 5x + 3y + 7z = 4 \ldots\ldots\ldots(2)$$

$$\therefore 121y - 11z = 33 \ldots\ldots\ldots(3)$$

Put $z = t$ in equation 3)

$$\therefore 121y - 11t = 33$$

$$\therefore 121y = 33 + 11t$$

Divide by 121

$$\boxed{\therefore y = \frac{3}{11} + \frac{1}{11}t}$$

Put $y = \frac{3}{11} + \frac{1}{11}t$ & $z = t$ in equation 1)

$$\therefore 5x + 3\left(\frac{3}{11} + \frac{1}{11}t\right) + 7t = 4$$

$$\therefore 5x + \frac{9}{11} + \frac{3}{11}t + 7t = 4$$

$$\therefore 5x + \frac{80}{11}t = 4 - \frac{9}{11}$$

$$\therefore 5x + \frac{80}{11}t = \frac{35}{11}$$

$$\therefore 5x = \frac{35}{11} - \frac{80}{11}t$$

Divide by 5

$$\boxed{\therefore x = \frac{7}{11} - \frac{16}{11}t}$$

3. Examine the following linear system of equation for consistency & solve it if consistent.

$\boldsymbol{4x - 2y + 6z = 8, x + y - 3z = -1, 15x - 3y + 9z = 21}$ **[S-23, W-23 4M]**

Sol: The given system of equation can be written as, $AX = B$

$$\therefore \begin{bmatrix} 4 & -2 & 6 \\ 1 & 1 & -3 \\ 15 & -3 & 9 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 8 \\ -1 \\ 21 \end{bmatrix}$$

$$R_1 \leftrightarrow R_2$$

$$\therefore \begin{bmatrix} 1 & 1 & -3 \\ 4 & -2 & 6 \\ 15 & -3 & 9 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -1 \\ 8 \\ 21 \end{bmatrix}$$

$$R_2 \to R_2 - 4 \times R_1, R_3 \to R_3 - 15 \times R_1$$

$R_2 - 4 \times R_1$	$R_3 - 15 \times R_1$
$4 - 4 \times 1 = 0$	$15 - 15 \times 1 = 0$
$-2 - 4 \times 1 = -6$	$-3 - 15 \times 1 = -18$
$6 - 4 \times (-3) = 18$	$9 - 15 \times (-3) = 54$
$8 - 4 \times (-1) = 12$	$21 - 15 \times (-1) = 36$

$$\therefore \begin{bmatrix} 1 & 1 & -3 \\ 0 & -6 & 18 \\ 0 & -18 & 54 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -1 \\ 12 \\ 36 \end{bmatrix}$$

$$R_3 \to R_3 - 3 \times R_2$$

$R_3 - 3 \times R_2$
$0 - 3 \times 0 = 0$
$-18 - 3 \times -6 = 0$
$54 - 3 \times 15 = 0$
$36 - 3 \times 12 = 0$

$$\therefore \begin{bmatrix} 1 & 1 & -3 \\ 0 & -6 & 18 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -1 \\ 12 \\ 0 \end{bmatrix} \dots\dots\dots(1)$$

$$\text{Here } \rho(A) = 2 \ \& \ \rho(K) = \left[\begin{array}{ccc|c} 1 & 1 & -3 & -1 \\ 0 & -6 & 18 & 12 \\ 0 & 0 & 0 & 0 \end{array}\right] = 2.$$

$$\therefore \rho(A) = 2 = \rho(K) = 2$$

This system is consistent.

Assign $n - r = 3 - 2 = 1$ parameter.

Consider $\boxed{z = t}$

From equation 1)

$$\therefore x + y - 3z = -1 \dots\dots\dots(2)$$

$$\therefore -6y + 18z = 12 \dots\dots\dots(3)$$

Put $z = t$ in equation 3)

$$\therefore -6y + 18(t) = 12$$

$$\therefore -6y = 12 - 18t$$

Divide by -6

$$\boxed{\therefore y = -2 + 3t}$$

Put $y = -2 + 3t, z = t$ in equation 1)

$$\therefore x + (-2 + 3t) - 3t = -1$$

$$\therefore x - 2 + 3t - 3t = -1$$

$$\therefore x - 2 = -1$$

$$\therefore x = -1 + 2$$

$$\boxed{\therefore x = 1}$$

4. Determine the consistency of set of equations. $x + 2y + z = 3, 2x + 3y + 2z = 5,$

$$\mathbf{3x - 5y + 5z = 2, 3x + 9y - z = 4.}$$

Sol: The given system of equation can be written as, $AX = B$

$$\therefore \begin{bmatrix} 1 & 2 & 1 \\ 2 & 3 & 2 \\ 3 & -5 & 5 \\ 3 & 9 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 3 \\ 5 \\ 2 \\ 4 \end{bmatrix}$$

$$R_2 \to R_2 - 2 \times R_1, R_3 \to R_3 - 3 \times R_1, R_4 \to R_4 - 3 \times R_1$$

$R_2 - 2 \times R_1$	$R_3 - 3 \times R_1$	$R_4 - 3 \times R_1$
$2 - 2 \times 1 = 0$	$3 - 3 \times 1 = 0$	$3 - 3 \times 1 = 0$
$3 - 2 \times 2 = -1$	$-5 - 3 \times 2 = -11$	$9 - 3 \times 2 = 3$
$2 - 2 \times 1 = 0$	$5 - 3 \times 1 = 2$	$-1 - 3 \times 1 = -4$
$5 - 2 \times 3 = -1$	$2 - 3 \times 3 = -7$	$4 - 3 \times 3 = -5$

$$\therefore \begin{bmatrix} 1 & 2 & 1 \\ 0 & -1 & 0 \\ 0 & -11 & 2 \\ 0 & 3 & -4 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 3 \\ -1 \\ -7 \\ -5 \end{bmatrix}$$

$$R_3 \to R_3 - 11 \times R_2, R_4 \to R_4 + 3 \times R_2$$

$R_3 - 11 \times R_2$	$R_4 + 3 \times R_2$
$0 - 11 \times 0 = 0$	$0 + 3 \times 0 = 0$
$-11 - 11 \times (-1) = 0$	$3 + 3 \times -1 = 0$
$2 - 11 \times 0 = 2$	$-4 + 3 \times 0 = -4$
$-7 - 11 \times (-1) = 4$	$-5 + 3 \times -1 = -8$

$$\therefore \begin{bmatrix} 1 & 2 & 1 \\ 0 & -1 & 0 \\ 0 & 0 & 2 \\ 0 & 0 & -4 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 3 \\ -1 \\ 4 \\ -8 \end{bmatrix}$$

$$R_4 \to R_4 + 2 \times R_3$$

$R_4 + 2 \times R_3$
$0 + 2 \times 0 = 0$
$0 + 2 \times 0 = 0$
$-4 + 2 \times 2 = 0$
$-8 + 2 \times 4 = 0$

$$\therefore \begin{bmatrix} 1 & 2 & 1 \\ 0 & -1 & 0 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 3 \\ -1 \\ 4 \\ 0 \end{bmatrix} \dots\dots\dots\dots 1)$$

Here $\rho(A) = 3$ & $\rho(K) = \left[\begin{array}{ccc|c} 1 & 2 & 1 & 3 \\ 0 & -1 & 0 & -1 \\ 0 & 0 & 2 & 4 \\ 0 & 0 & 0 & 0 \end{array}\right] = 3$. Number of unknowns $= 3$.

$$\therefore \rho(A) = \rho(K) = n = 3$$

System is consistent & has a unique solution.

From equation 1), we get.

$$x + 2y + z = 3 \dots\dots 2)$$

$$-y = -1 \Rightarrow \boxed{y = 1}$$

$$2z = 4 \Rightarrow z = \frac{4}{2} \quad \boxed{\therefore z = 2}$$

Put $y = 1, z = 2$ in equation 2)

$$x + 2(1) + 2 = 3$$

$$\therefore x + 4 = 3$$

$$\therefore x = 3 - 4 \Rightarrow \boxed{x = -1}$$

The required solution is $(-1, 1, 2)$.

5. **Check the consistency of set of equations & solve. $2x - 3y + 5z = 1, 3x + y - z = 2$, $x + 4y - 6z = 1$ (Dec-2023) 6Marks**

Sol: The given system of equation can be written as, $AX = B$

$$\begin{bmatrix} 2 & -3 & 5 \\ 3 & 1 & -1 \\ 1 & 4 & -6 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$$

$$R_1 \leftrightarrow R_3$$

$$\begin{bmatrix} 1 & 4 & -6 \\ 3 & 1 & -1 \\ 2 & -3 & 5 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$$

$$R_2 \to R_2 - 3 \times R_1, R_3 \to R_3 - 2 \times R_1$$

$R_2 - 3 \times R_1$	$R_3 - 2 \times R_1$
$3 - 3 \times 1 = 0$	$2 - 2 \times 1 = 0$
$1 - 3 \times 4 = -11$	$-3 - 2 \times 4 = -11$
$-1 - 3 \times -6 = 17$	$5 - 2 \times -6 = 17$
$2 - 3 \times 1 = -1$	$1 - 2 \times 1 = -1$

$$\begin{bmatrix} 1 & 4 & -6 \\ 0 & -11 & 17 \\ 0 & -11 & 17 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \\ -1 \end{bmatrix}$$

$$R_3 \to R_3 - R_2$$

$R_3 - R_2$

$0-0=0$
$-11-(-11)=0$
$17-17=0$
$-1-(-1)=0$

$$\begin{bmatrix}1 & 4 & -6\\0 & -11 & 17\\0 & 0 & 0\end{bmatrix}\begin{bmatrix}x\\y\\z\end{bmatrix}=\begin{bmatrix}1\\-1\\0\end{bmatrix}\ldots\ldots.1)$$

Here $\rho(A)=2$ & $\rho(K)=\left[\begin{array}{ccc|c}1 & 4 & -6 & 1\\0 & -11 & 17 & -1\\0 & 0 & 0 & 0\end{array}\right]=2$

$$\therefore \rho(A)=2=\rho(K)=2<n=3$$

System is consistent & has infinite solution.

From equation 1), we get.

$$x+4y-6z=1\ldots\ldots 2)$$

$$-11y+17z=-1\ldots..3)$$

Consider $\boxed{z=t}$

From equation 3)

$$\therefore -11y+17t=-1$$

$$\therefore -11y=-1-17t$$

$$\therefore 11y=1+17t$$

$$\boxed{\therefore y=\frac{1}{11}+\frac{17}{11}t}$$

From equation 2)

$$x+4\left[\frac{1}{11}+\frac{17}{11}t\right]-6t=1$$

$$\therefore x+\frac{4}{11}+\frac{68}{11}t-6t=1$$

$$\therefore x+\frac{4}{11}+\frac{2}{11}t=1$$

$$\therefore x=1-\frac{4}{11}-\frac{2}{11}t$$

$$\boxed{\therefore x=\frac{7}{11}-\frac{2}{11}t}$$

The required solution is $\left(\frac{7}{11}-\frac{2}{11}t,\frac{1}{11}+\frac{17}{11}t,t\right)$

6. For what value of k is the following system of equations is consistent & hence solve for them

$$x+y+z=1, x+2y+4z=k, x+4y+10z=k^2$$

Sol: The given system of equation can be written as, $AX=B$

$$\begin{bmatrix}1 & 1 & 1\\1 & 2 & 4\\1 & 4 & 10\end{bmatrix}\begin{bmatrix}x\\y\\z\end{bmatrix}=\begin{bmatrix}1\\k\\k^2\end{bmatrix}$$

$$R_2\to R_2-R_1, R_3\to R_3-R_1$$

$R_2 - R_1$	$R_3 - R_1$
$1 - 1 = 0$	$1 - 1 = 0$
$2 - 1 = 1$	$4 - 1 = 3$
$4 - 1 = 3$	$10 - 1 = 9$
$k - 1 = k - 1$	$k^2 - 1 = k^2 - 1$

$$\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 3 \\ 0 & 3 & 9 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ k-1 \\ k^2-1 \end{bmatrix}$$

$$R_3 \to R_3 - 3 \times R_2$$

$R_3 - 3 \times R_2$
$0 - 3 \times 0 = 0$
$3 - 3 \times 1 = 0$
$9 - 3 \times 3 = 0$
$k^2 - 1 - 3 \times (k-1) = k^2 - 1 - 3k + 3 = k^2 - 3k + 2$

$$\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 3 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ k-1 \\ k^2-3k+2 \end{bmatrix} \ldots\ldots 1)$$

Here $\rho(A) = 2$

The rank of k also be 2 if $k^2 - 3k + 2 = 0$, on solving we get $k = 1,2$.

The ranks of A & K will be equal for $k = 1,2$.

The given system equation will be consistent for $k = 1,2$.

We get $\rho(A) = 2 = \rho(K) = 2 < n = 3$

Hence the system have infinite no. of solutions for $k = 1,2$.

For $k = 1$.

Equation 1) becomes;

$$\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 3 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 1-1 \\ 1^2-3(1)+2 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 3 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

$$x + y + z = 1$$

$$y + 3z = 0$$

$$\text{Put } \boxed{z = t}$$

$$y + 3t = 0 \to \boxed{y = -3t}$$

$$\text{Put } y = -3t \ \& \ z = t$$

$$x + (-3t) + t = 1$$

$$x - 2t = 1 \to \boxed{x = 1 + 2t}$$

The required solution is $(1 + 2t, -3t, t)$

For $k = 2.$

Equation 1) becomes;

$$\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 3 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 2-1 \\ 2^2 - 3(2) + 2 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 3 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$$

$$x + y + z = 1$$

$$y + 3z = 1$$

$$\text{Put } \boxed{z = t_1}$$

$$y + 3t_1 = 1 \rightarrow \boxed{y = 1 - 3t_1}$$

$$\text{Put } y = 1 - 3t_1 \;\&\; z = t_1$$

$$x + 1 - 3t_1 + t_1 = 1$$

$$x + 1 - 2t_1 = 1 \rightarrow x = 1 + 2t_1 - 1 \rightarrow \boxed{x = 2t_1}$$

The required solution is $(2t_1, 1 - 3t_1, t_1)$

- **HOMEWORK:**

Check the consistency of set of equations & solve.

1. $5x + 3y + 7z = 4, 3x + 26y + 2z = 9, 7x + 2y + 10z = 5$

2. $4x - 2y + 6z = 8, x + y - 3z = -1, 15x - 3y + 9z = 21$

3. $x - 2y + 3t = 2, 2x + y + z + t = 4, 4x - 3y + z + 7t = 8$

4. $x_1 + x_2 + x_3 = 3, 2x_1 - x_2 + 3x_3 = 1, 4x_1 + x_2 + 5x_3 = 2, 3x_1 - 2x_2 + x_3 = 4$

5. $x_1 + x_2 + x_3 = 6, x_1 - x_2 + 2x_3 = 5, 3x_1 + x_2 + x_3 = 8, 2x_1 - 2x_2 + 3x_3 = 7$

6. $5x + 3y + 7z = 4, 3x + 26y + 2z = 9, 7x + 2y + 10z = 5$

UNIT 2

MATRICES (20 MARKS)

Topic Content: Inverse of a matrix by Gauss-Jordan method; Rank of a matrix; Normal form of a matrix; **Consistency of non- homogeneous and homogeneous system of linear equations**; Eigen values and Eigen vectors; Properties of Eigen values and Eigen vectors (without proofs); Cayley-Hamilton's theorem (without proof) and its applications.

MR SUDHIR S DESAI

Course Outcome: After completion of this course, students will be able to

CO2: Implement matrix concept to solve real life problems.

❖ Consistency of homogeneous system of linear equations:

Let us consider the homogeneous system of m linear equations in n unknowns $x_1, x_2, x_3, \dots \dots \dots, x_n$.

$$\left.\begin{array}{l} a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + \cdots + a_{1n}x_n = 0, \\ a_{21}x_1 + a_{22}x_2 + a_{23}x_3 + \cdots + a_{2n}x_n = 0, \\ \vdots \qquad \vdots \qquad \vdots \qquad \qquad \vdots \qquad \vdots \\ a_{m1}x_1 + a_{m2}x_2 + a_{m3}x_3 + \cdots + a_{mn}x_n = 0 \end{array}\right\} 1)$$

To check the consistency (i.e. if the system of equation has at least one solution) we find the rank of matrices.

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} & \dots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \dots & a_{2n} \\ \vdots & \vdots & \vdots & \dots & \vdots \\ a_{m1} & a_{m2} & a_{m3} & \dots & a_{mn} \end{bmatrix}$$

Here A is called coefficient matrix of equation 1)

We find the rank of matrix A by reducing them to triangular form by applying elementary row transformations.

➢ <u>Conditions To Check Consistency:</u>

Case I) If $\rho(A) = n$ (**Number of unknowns**) then the system of equation 1) will have only a trivial solution. $x_1 = x_2 = x_3 = \cdots \dots \dots = x_n = 0$

Case II) If $\rho(A) < n$ (**Number of unknowns**) then the system of equation 1) will have infinite number of solution (non-trivial solution) & assign $n - r$ parameters to variables.

<u>**Note:**</u> **The necessary & sufficient condition for the system of equations 1) to have non trivial solution is that determinant of coefficient matrix is zero.**

- **Examples:**

1. **Solve the equations.** $\boldsymbol{x+2y+3z=0, 3x+4y+4z=0, 7x+10y+12z=0}$.

Sol: The given system of equation can be written as, $AX=B$

$$\therefore \begin{bmatrix} 1 & 2 & 3 \\ 3 & 4 & 4 \\ 7 & 10 & 12 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix}=\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$R_2 \to R_2 - 3\times R_1, R_3 \to R_3 - 7\times R_1$$

$\boldsymbol{R_2-3\times R_1}$	$\boldsymbol{R_3-7\times R_1}$
$3-3\times 1=0$	$7-7\times 1=0$
$4-3\times 2=-2$	$10-7\times 2=-4$
$4-3\times 3=-5$	$12-7\times 3=-9$

$$\therefore \begin{bmatrix} 1 & 2 & 3 \\ 0 & -2 & -5 \\ 0 & -4 & -9 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix}=\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$R_3 \to R_3 - 2\times R_2$$

$\boldsymbol{R_3-2\times R_2}$
$0-2\times 0=0$
$-4-2\times(-2)=0$
$-9-2\times(-5)=1$

$$\therefore \begin{bmatrix} 1 & 2 & 3 \\ 0 & -2 & -5 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix}=\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Here $\rho(A)=3$ &. Number of unknowns $=3$.

$$\therefore \rho(A)=n=3$$

System will have a trivial solution.

$$\boxed{\boldsymbol{x=y=z=0}.}$$

2. **Solve the equations.** $\boldsymbol{x+3y+2z=0, 2x-y+3z=0, 3x-5y+4z=0}$.

Sol: The given system of equation can be written as, $AX=B$

$$\therefore \begin{bmatrix} 1 & 3 & 2 \\ 2 & -1 & 3 \\ 3 & -5 & 4 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix}=\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$R_2 \to R_2 - 2\times R_1, R_3 \to R_3 - 3\times R_1$$

$\boldsymbol{R_2-2\times R_1}$	$\boldsymbol{R_3-3\times R_1}$
$2-2\times 1=0$	$3-3\times 1=0$
$-1-2\times 3=-7$	$-5-3\times 3=-14$
$3-2\times 2=-1$	$4-3\times 2=-2$

$$\therefore \begin{bmatrix} 1 & 3 & 2 \\ 0 & -7 & -1 \\ 0 & -14 & -2 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix}=\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$R_3 \to R_3 - 2\times R_2$$

$\boldsymbol{R_3-2\times R_2}$

$0-2\times 0=0$
$-14-2\times(-7)=0$
$-2-2\times(-1)=0$

$$\therefore \begin{bmatrix}1 & 3 & 2\\ 0 & -7 & -1\\ 0 & 0 & 0\end{bmatrix}\begin{bmatrix}x\\ y\\ z\end{bmatrix}=\begin{bmatrix}0\\ 0\\ 0\end{bmatrix}\ldots\ldots\ldots 1)$$

Here $\rho(A)=2<n=3$.

System will have infinite number of solutions.

Assign $n-r=3-2=1$ parameter.

Consider $\boxed{z=t}$

From equation 1), we get.

$$x+3y+2z=0\ldots\ldots 2)$$

$$-7y-z=0\ldots..3)$$

$$\therefore -7y-t=0\Rightarrow 7y+t=0$$

$$\therefore 7y=-t\Longrightarrow \boxed{y=\frac{-t}{7}}$$

Put $y=\frac{-t}{7}, z=t$ in equation 2)

$$\therefore x+3\left(\frac{-t}{7}\right)+2t=0$$

$$\therefore x-\frac{3t}{7}+2t=0$$

$$\therefore x+\frac{11t}{7}=0\Longrightarrow \boxed{x=-\frac{11t}{7}}$$

The required solution is $\left(-\frac{11t}{7},-\frac{t}{7},t\right)$

3. **Find the value of k so that the equations $x+y+3z=0, 4x+3y+kz=0, 2x+y+2z=0$ have non-trivial solution.**

Sol: The determinant of coefficient matrix should be zero for the given system of equations to have non-trivial solution.

$$\therefore \begin{vmatrix}1 & 1 & 3\\ 4 & 3 & k\\ 2 & 1 & 2\end{vmatrix}=0$$

$$\therefore (1)\begin{vmatrix}3 & k\\ 1 & 2\end{vmatrix}-(1)\begin{vmatrix}4 & k\\ 2 & 2\end{vmatrix}+(3)\begin{vmatrix}4 & 3\\ 2 & 1\end{vmatrix}=0$$

$$\therefore 1.[6-k]-1.[8-2k]+3.[4-6]=0$$

$$\therefore 6-k-8+2k+12-18=0$$

$$\therefore k-8=0$$

$$\boxed{\therefore k=8}$$

For $k=8$ the given system of equations will have non-trivial solutions.

- **HOMEWORK:**

Solve the equations $4x+2y+z+3w=0, 6x+3y+4z+7w=0, 2x+y+w=0$

UNIT 2

MATRICES (20 MARKS)

Topic Content: Inverse of a matrix by Gauss-Jordan method; Rank of a matrix; Normal form of a matrix; Consistency of non- homogeneous and homogeneous system of linear equations; Eigen values and **Eigen vectors; Properties of Eigen values and Eigen vectors (without proofs)**

MR SUDHIR S DESAI

Course Outcome: After completion of this course, students will be able to

CO2: Implement matrix concept to solve real life problems.

❖ Eigen values and Eigen vectors:

- **Characteristic Equation:**

  Consider A be the n rowed square matrix, λ be any scalar and I is be the identity matrix of same order as A. Then matrix $[A-\lambda I]$ is called as characteristic matrix.

  The determinant $|A-\lambda I|$ is called as characteristic polynomial of A & equation $|A-\lambda I|=0$ is called as characteristic equation.

- **Eigen Values:**

  The roots of characteristic equation $|A-\lambda I|=0$ is called as Eigen values.

- **Eigen Vectors:**

  A non-zero vector that only gets scaled (not rotated) when a linear transformation is applied to it.

- **Methods of finding characteristic equation:**

  a) The characteristic equation of a 3 x 3 matrix is $\lambda^3 - s_1\lambda^2 + s_2\lambda - |A| = 0$

  Where, $s_1 =$ Sum of main diagonal elements.

  $s_2 =$ Sum of minors of main diagonal elements.

  b) The characteristic equation of a 2 x 2 matrix is $\lambda^2 - s_1\lambda + |A| = 0$

  Where, $s_1 =$ Sum of main diagonal elements.

❖ Properties of Eigen values & Eigen vectors:

I. The Eigen values of a square matrix A & it's transpose A^T are same.

II. The Eigen values of a triangular matrix are just its diagonal elements.

If $A = \begin{bmatrix} 3 & 1 & 4 \\ 0 & 2 & 6 \\ 0 & 0 & 5 \end{bmatrix}$ then Eigen values are $\lambda = 3,2,5$

III. The Eigen values of an idempotent matrix are either zero or unity.

The square matrix A is said to be idempotent if $A^2 = A$ then $\lambda = 0,1$

IV. The sum of Eigen values of matrix A is equal to the sum of its diagonal elements.

V. The product of Eigen values of a square matrix A is equal to the determinant of the matrix A.

VI. If λ is the Eigen value of matrix A, then $\frac{1}{\lambda}$ will be the Eigen values of A^{-1}

VII. If λ is the Eigen value of the orthogonal matrix A, then $\frac{1}{\lambda}$ will also be the Eigen values of A.

VIII. If λ is the Eigen value of matrix A are $\lambda_1, \lambda_2, \ldots\ldots, \lambda_n$ then the Eigen values of the matrix A^m are $\lambda_1{}^m, \lambda_2{}^m, \ldots\ldots, \lambda_n{}^m$. m being a positive integer.

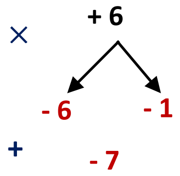
- **Examples:**

1. Find the Eigen values for the following matrix. $A = \begin{bmatrix} \mathbf{5} & \mathbf{4} \\ \mathbf{1} & \mathbf{2} \end{bmatrix}$ **[S-23, W-23 2M]**

Sol: Given $A = \begin{bmatrix} 5 & 4 \\ 1 & 2 \end{bmatrix}$

The characteristic equation is $|A - \lambda I| = 0$

$\therefore \begin{vmatrix} 5-\lambda & 4 \\ 1 & 2-\lambda \end{vmatrix} = 0$



$\therefore (5-\lambda)(2-\lambda) - (4)(1) = 0$

$\therefore 10 - 5\lambda - 2\lambda + \lambda^2 - 4 = 0$

$\therefore \lambda^2 - 7\lambda + 6 = 0$

$\therefore (\lambda - 6)(\lambda - 1) = 0$

$\therefore \lambda - 6 = 0 \;\&\; \lambda - 1 = 0$

$\boxed{\therefore\ \boldsymbol{\lambda = 6 \;\&\; \lambda = 1}}$

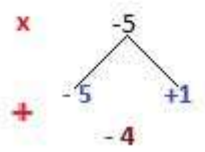
These are required Eigen Values.

2. Find the Eigen values & Eigen vectors for the following matrix. $A = \begin{bmatrix} \mathbf{1} & \mathbf{2} \\ \mathbf{4} & \mathbf{3} \end{bmatrix}$

Sol: Given $A = \begin{bmatrix} 1 & 2 \\ 4 & 3 \end{bmatrix}$

The characteristic equation is $|A - \lambda I| = 0$

$\therefore \begin{vmatrix} 1-\lambda & 2 \\ 4 & 3-\lambda \end{vmatrix} = 0$



$\therefore (1-\lambda)(3-\lambda) - (2)(4) = 0$

$\therefore 3 - \lambda - 3\lambda + \lambda^2 - 8 = 0$

$\therefore \lambda^2 - 4\lambda - 5 = 0$

$\therefore (\lambda - 5)(\lambda + 1) = 0$

$\therefore \lambda - 5 = 0 \;\&\; \lambda + 1 = 0$

$\boxed{\therefore\ \boldsymbol{\lambda = 5 \;\&\; \lambda = -1}}$

These are required Eigen Values.

To find Eigen Vectors:

Consider $[A - \lambda I]X = 0$

$\therefore \begin{bmatrix} 1-\lambda & 2 \\ 4 & 3-\lambda \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \ldots\ldots 1)$

For $\lambda = 5$:

Put $\lambda = 5$ in equation 1)

$\therefore \begin{bmatrix} 1-5 & 2 \\ 4 & 3-5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$\therefore \begin{bmatrix} -4 & 2 \\ 4 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$\therefore \boldsymbol{-4x_1 + 2x_2 = 0 \ldots\ldots 2)}$

& $\mathbf{4x_1 - 2x_2 = 0 \ldots \ldots \ldots 3)}$

Consider, $4x_1 - 2x_2 = 0$

Divide by 2

$\therefore 2x_1 - x_2 = 0$

$\therefore 2x_1 = x_2$

$\therefore \dfrac{x_1}{1} = \dfrac{x_2}{2}$

$\therefore x_1 = 1$ & $x_2 = 2$

$\therefore$ For $\lambda = 5$ the required Eigen vector is $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$

For $\lambda = -1$:

Put $\lambda = -1$ in equation 1)

$$\therefore \begin{bmatrix} 1-(-1) & 2 \\ 4 & 3-(-1) \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\therefore \begin{bmatrix} 2 & 2 \\ 4 & 4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$\therefore \mathbf{2x_1 + 2x_2 = 0 \ldots \ldots .4)}$

& $\mathbf{4x_1 + 4x_2 = 0 \ldots \ldots .5)}$

Consider, $4x_1 + 4x_2 = 0$

Divide by 4

$\therefore x_1 + x_2 = 0$

$\therefore x_1 = -x_2$

$\therefore \dfrac{x_1}{-1} = \dfrac{x_2}{1}$

$\therefore x_1 = -1$ & $x_2 = 1$

$\therefore$ For $\lambda = -1$ the required Eigen vector is $\begin{bmatrix} -1 \\ 1 \end{bmatrix}$

3. Find the Eigen values & Eigen vectors for the following matrix. $A = \begin{bmatrix} 8 & -6 & 2 \\ -6 & 7 & -4 \\ 2 & -4 & 3 \end{bmatrix}$

Sol: Given $A = \begin{bmatrix} 8 & -6 & 2 \\ -6 & 7 & -4 \\ 2 & -4 & 3 \end{bmatrix}$

The characteristic equation is $|A - \lambda I| = 0$

$$\therefore \begin{vmatrix} 8-\lambda & -6 & 2 \\ -6 & 7-\lambda & -4 \\ 2 & -4 & 3-\lambda \end{vmatrix} = 0$$

$\therefore \lambda^3 - s_1\lambda^2 + s_2\lambda - |A| = 0 \ldots \ldots 1)$

$\therefore s_1$ = Sum of main diagonal elements of matrix A.

$\therefore s_1 = 8 + 7 + 3 \Rightarrow \boxed{s_1 = 18}$

s_2 = Sum of minors of main diagonal elements of matrix A.

$$\therefore s_2 = \begin{vmatrix} 7 & -4 \\ -4 & 3 \end{vmatrix} + \begin{vmatrix} -6 & -4 \\ 2 & 3 \end{vmatrix} + \begin{vmatrix} -6 & 7 \\ 2 & -4 \end{vmatrix}$$

$\therefore s_2 = 21 - 16 + (-18) - (-8) + 24 - 14 \Rightarrow \boxed{s_2 = 45}$

$$\& \ |A| = \begin{vmatrix} 8 & -6 & 2 \\ -6 & 7 & -4 \\ 2 & -4 & 3 \end{vmatrix}$$

$$\therefore |A| = (8)\begin{vmatrix} 7 & -4 \\ -4 & 3 \end{vmatrix} - (-6)\begin{vmatrix} -6 & -4 \\ 2 & 3 \end{vmatrix} + (2)\begin{vmatrix} -6 & 7 \\ 2 & -4 \end{vmatrix}$$

$\therefore |A| = 8[21 - 16] + 6[-18 - (-8)] + 2[24 - 14]$

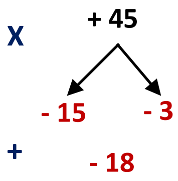
$\boxed{\therefore |A| = 0}$

Equation 1) becomes;

$\therefore \lambda^3 - 18\lambda^2 + 45\lambda - 0 = 0$

$\therefore \lambda^3 - 18\lambda^2 + 45\lambda = 0$

Taking λ comman.



$\therefore \lambda[\lambda^2 - 18\lambda + 45] = 0$

$\therefore \lambda(\lambda - 15)(\lambda - 3) = 0$

$\therefore \lambda = 0, \lambda - 15 = 0, \lambda - 3 = 0$

$\boxed{\therefore \lambda = 0, \lambda = 15, \lambda = 3}$

These are required Eigen Values.

To find Eigen Vectors:

Consider $[A - \lambda I]X = 0$

$$\therefore \begin{bmatrix} 8-\lambda & -6 & 2 \\ -6 & 7-\lambda & -4 \\ 2 & -4 & 3-\lambda \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \ldots\ldots 1)$$

For $\lambda = 0$:

Put $\lambda = 0$ in equation 1)

$$\therefore \begin{bmatrix} 8-0 & -6 & 2 \\ -6 & 7-0 & -4 \\ 2 & -4 & 3-0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\therefore \begin{bmatrix} 8 & -6 & 2 \\ -6 & 7 & -4 \\ 2 & -4 & 3 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Writing equations;

$8x_1 - 6x_2 + 2x_3 = 0$

Divide by 2

$\therefore 4x_1 - 3x_2 + x_3 = 0 \ldots\ldots 2)$

$-6x_1 + 7x_2 - 4x_3 = 0 \ldots\ldots 3)$

$2x_1 - 4x_2 + 3x_3 = 0 \ldots\ldots 4)$

Solving 2) & 3) Using Crammers Rule,

$$\frac{x_1}{\begin{vmatrix} -3 & 1 \\ 7 & -4 \end{vmatrix}} = \frac{-x_2}{\begin{vmatrix} 4 & 1 \\ -6 & -4 \end{vmatrix}} = \frac{x_3}{\begin{vmatrix} 4 & -3 \\ -6 & 7 \end{vmatrix}}$$

$$\therefore \frac{x_1}{12 - 7} = \frac{-x_2}{-16 - (-6)} = \frac{x_3}{28 - 18}$$

$$\therefore \frac{x_1}{5} = \frac{-x_2}{-10} = \frac{x_3}{10}$$

Divide by 5

$$\therefore \frac{x_1}{1} = \frac{x_2}{2} = \frac{x_3}{2}$$

$\therefore \lambda = 0$ **Corresponding Eigen vector is** $\begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}$

For $\lambda = 15$:

Put $\lambda = 15$ in equation 1)

$$\therefore \begin{bmatrix} 8-15 & -6 & 2 \\ -6 & 7-15 & -4 \\ 2 & -4 & 3-15 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\therefore \begin{bmatrix} -7 & -6 & 2 \\ -6 & -8 & -4 \\ 2 & -4 & -12 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Writing equations;

$$\mathbf{-7x_1 - 6x_2 + 2x_3 = 0 \ldots\ldots 2)}$$

$$-6x_1 - 8x_2 - 4x_3 = 0$$

Divide by – 2

$$\mathbf{3x_1 + 4x_2 + 2x_3 = 0 \ldots\ldots 3)}$$

$$2x_1 - 4x_2 - 12x_3 = 0$$

Divide by 2

$$\mathbf{x_1 - 2x_2 - 6x_3 = 0 \ldots\ldots 4)}$$

Solving 3) & 4) Using Crammers Rule,

$$\frac{x_1}{\begin{vmatrix} 4 & 2 \\ -2 & -6 \end{vmatrix}} = \frac{-x_2}{\begin{vmatrix} 3 & 2 \\ 1 & -6 \end{vmatrix}} = \frac{x_3}{\begin{vmatrix} 3 & 4 \\ 1 & -2 \end{vmatrix}}$$

$$\therefore \frac{x_1}{-24-(-4)} = \frac{-x_2}{-18-2} = \frac{x_3}{-6-4}$$

$$\therefore \frac{x_1}{-20} = \frac{-x_2}{-20} = \frac{x_3}{-10}$$

Divide by – 10

$$\therefore \frac{x_1}{2} = \frac{-x_2}{2} = \frac{x_3}{1}$$

$$\therefore \frac{x_1}{2} = \frac{x_2}{-2} = \frac{x_3}{1}$$

$\therefore \lambda = 15$ **corresponding Eigen vector is** $\begin{bmatrix} 2 \\ -2 \\ 1 \end{bmatrix}$

For $\lambda = 3$:

Put $\lambda = 3$ in equation 1)

$$\therefore \begin{bmatrix} 8-3 & -6 & 2 \\ -6 & 7-3 & -4 \\ 2 & -4 & 3-3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\therefore \begin{bmatrix} 5 & -6 & 2 \\ -6 & 4 & -4 \\ 2 & -4 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Writing equations;

$$\mathbf{5x_1 - 6x_2 + 2x_3 = 0 \ldots\ldots 2)}$$

$$-6x_1 + 4x_2 - 4x_3 = 0$$

Divide by – 2

$$\mathbf{3x_1 - 2x_2 + 2x_3 = 0 \ldots\ldots 3)}$$

$$2x_1 - 4x_2 + 0x_3 = 0$$

Divide by 2

$$\mathbf{x_1 - 2x_2 - 0x_3 = 0 \ldots\ldots 4)}$$

Solving 3) & 4) Using Crammers Rule,

$$\frac{x_1}{\begin{vmatrix} -2 & 2 \\ -2 & 0 \end{vmatrix}} = \frac{-x_2}{\begin{vmatrix} 3 & 2 \\ 1 & 0 \end{vmatrix}} = \frac{x_3}{\begin{vmatrix} 3 & -2 \\ 1 & -2 \end{vmatrix}}$$

$$\therefore \frac{x_1}{0-(-2)} = \frac{-x_2}{0-2} = \frac{x_3}{-6-(-2)}$$

$$\therefore \frac{x_1}{2} = \frac{-x_2}{-2} = \frac{x_3}{-4}$$

Divide by 2

$$\therefore \frac{x_1}{1} = \frac{-x_2}{-1} = \frac{x_3}{-2}$$

$$\therefore \frac{x_1}{1} = \frac{x_2}{1} = \frac{x_3}{-2}$$

$\therefore \lambda = 3$ **corresponding Eigen vector is** $\begin{bmatrix} \mathbf{1} \\ \mathbf{1} \\ \mathbf{-2} \end{bmatrix}$

4. **Find the Eigen values & Eigen vectors for the following matrix.** $A = \begin{bmatrix} \mathbf{1} & \mathbf{0} & \mathbf{-4} \\ \mathbf{0} & \mathbf{5} & \mathbf{4} \\ \mathbf{-4} & \mathbf{4} & \mathbf{3} \end{bmatrix}$

Sol: Given $A = \begin{bmatrix} 1 & 0 & -4 \\ 0 & 5 & 4 \\ -4 & 4 & 3 \end{bmatrix}$

The characteristic equation is $|A - \lambda I| = 0$

$$\therefore \begin{vmatrix} 1-\lambda & 0 & -4 \\ 0 & 5-\lambda & 4 \\ -4 & 4 & 3-\lambda \end{vmatrix} = 0$$

$$\therefore \lambda^3 - s_1\lambda^2 + s_2\lambda - |A| = 0 \ldots\ldots 1)$$

$\therefore s_1 =$ Sum of main diagonal elements of matrix A.

$$\therefore s_1 = 1 + 5 + 3 \Rightarrow \boxed{s_1 = 9}$$

$s_2 =$ Sum of minors of main diagonal elements of matrix A.

$$\therefore s_2 = \begin{vmatrix} 5 & 4 \\ 4 & 3 \end{vmatrix} + \begin{vmatrix} 1 & -4 \\ -4 & 3 \end{vmatrix} + \begin{vmatrix} 1 & 0 \\ 0 & 5 \end{vmatrix}$$

$$\therefore s_2 = 15 - 16 + 3 - 16 + 5 - 0 \Rightarrow \boxed{s_2 = -9}$$

$$\& \; |A| = \begin{vmatrix} 1 & 0 & -4 \\ 0 & 5 & 4 \\ -4 & 4 & 3 \end{vmatrix}$$

$$\therefore |A| = (1)\begin{vmatrix} 5 & 4 \\ 4 & 3 \end{vmatrix} - (0)\begin{vmatrix} 0 & 4 \\ -4 & 3 \end{vmatrix} + (-4)\begin{vmatrix} 0 & 5 \\ -4 & 4 \end{vmatrix}$$

$$\therefore |A| = 1[15 - 16] + 0 - 4[0 - (-20)]$$

$$\boxed{\therefore |A| = -81}$$

Equation 1) becomes;

$$\therefore \lambda^3 - 9\lambda^2 - 9\lambda + 81 = 0$$

Here $\lambda = 3$ is one of its Eigen value.

By Synthetic Division Method.

$$\begin{array}{c|cccc} & \lambda^3 & \lambda^2 & \lambda & C \\ 3 & 1 & -9 & -9 & 81 \\ & & 3 & -18 & -81 \\ \hline & 1 & -6 & -27 & \boxed{0} \\ & \lambda^2 & \lambda & C & \end{array}$$

X: -27 → -9, 3

+: -6

$$\therefore (\lambda - 3)(\lambda^2 - 6\lambda - 27) = 0$$

$$\therefore (\lambda - 3)(\lambda - 9)(\lambda + 3) = 0$$

$$\boxed{\therefore \lambda = 3, 9, -3}$$

These are required Eigen values.

To find Eigen vectors:

Consider $[A - \lambda I]X = 0$

$$\therefore \begin{bmatrix} 1-\lambda & 0 & -4 \\ 0 & 5-\lambda & 4 \\ -4 & 4 & 3-\lambda \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \ldots \ldots 2)$$

For $\lambda = 3$:

Put $\lambda = 3$ in equation 1)

$$\therefore \begin{bmatrix} 1-3 & 0 & -4 \\ 0 & 5-3 & 4 \\ -4 & 4 & 3-3 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\therefore \begin{bmatrix} -2 & 0 & -4 \\ 0 & 2 & 4 \\ -4 & 4 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Writing equations;

$$-2x_1 + 0x_2 - 4x_3 = 0$$

Divide by 2

$$\therefore -x_1 + 0x_2 - 2x_3 = 0 \ldots ..3)$$

$$0x_1 + 2x_2 + 4x_3 = 0$$

Divide by 2

$$\therefore 0x_1 + x_2 + 2x_3 = 0 \ldots \ldots 4)$$

$$-4x_1 + 4x_2 + 0x_3 = 0$$

Divide by 4

$\therefore \mathbf{-x_1 + 2x_2 + 0x_3 = 0 \ldots \ldots 5)}$

Solving 3) & 4) Using Crammers Rule,

$$\frac{x_1}{\begin{vmatrix} 0 & -2 \\ 1 & 2 \end{vmatrix}} = \frac{-x_2}{\begin{vmatrix} -1 & -2 \\ 0 & 2 \end{vmatrix}} = \frac{x_3}{\begin{vmatrix} -1 & 0 \\ 0 & 1 \end{vmatrix}}$$

$$\therefore \frac{x_1}{0-(-2)} = \frac{-x_2}{-2-0} = \frac{x_3}{-1-0}$$

$$\therefore \frac{x_1}{2} = \frac{-x_2}{-2} = \frac{x_3}{-1}$$

$$\therefore \frac{x_1}{2} = \frac{x_2}{2} = \frac{x_3}{-1}$$

$\therefore \lambda = 3$ **corresponding Eigen vector is** $\begin{bmatrix} \mathbf{2} \\ \mathbf{2} \\ \mathbf{-1} \end{bmatrix}$

For $\lambda = 9$:

Put $\lambda = 9$ in equation 1)

$$\therefore \begin{bmatrix} 1-9 & 0 & -4 \\ 0 & 5-9 & 4 \\ -4 & 4 & 3-9 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\therefore \begin{bmatrix} -8 & 0 & -4 \\ 0 & -4 & 4 \\ -4 & 4 & -6 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Writing equations;

$-8x_1 + 0x_2 - 4x_3 = 0$

Divide by 4

$\therefore \mathbf{-2x_1 + 0x_2 - x_3 = 0 \ldots \ldots 3)}$

$0x_1 - 4x_2 + 4x_3 = 0$

Divide by 4

$\therefore \mathbf{0x_1 - x_2 + x_3 = 0 \ldots \ldots 4)}$

$-4x_1 + 4x_2 - 6x_3 = 0$

Divide by 2

$\therefore \mathbf{-2x_1 + 2x_2 - 3x_3 = 0 \ldots \ldots 5)}$

Solving 3) & 4) Using Crammers Rule,

$$\frac{x_1}{\begin{vmatrix} 0 & -1 \\ -1 & 1 \end{vmatrix}} = \frac{-x_2}{\begin{vmatrix} -2 & -1 \\ 0 & 1 \end{vmatrix}} = \frac{x_3}{\begin{vmatrix} -2 & 0 \\ 0 & -1 \end{vmatrix}}$$

$$\therefore \frac{x_1}{0-1} = \frac{-x_2}{-2-0} = \frac{x_3}{2-0}$$

$$\therefore \frac{x_1}{-1} = \frac{-x_2}{-2} = \frac{x_3}{2}$$

$$\therefore \frac{x_1}{-1} = \frac{x_2}{2} = \frac{x_3}{2}$$

$\therefore \lambda = 9$ **Corresponding Eigen vector is** $\begin{bmatrix} \mathbf{-1} \\ \mathbf{2} \\ \mathbf{2} \end{bmatrix}$

For $\lambda = -3$:

Put $\lambda = -3$ in equation 1)

$$\therefore \begin{bmatrix} 1-(-3) & 0 & -4 \\ 0 & 5-(-3) & 4 \\ -4 & 4 & 3-(-3) \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\therefore \begin{bmatrix} 4 & 0 & -4 \\ 0 & 8 & 4 \\ -4 & 4 & 6 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Writing equations;

$4x_1 + 0x_2 - 4x_3 = 0$

Divide by 4

$\therefore \mathbf{x_1 + 0x_2 - x_3 = 0 \ldots\ldots 3)}$

$0x_1 + 8x_2 + 4x_3 = 0$

Divide by 4

$\therefore \mathbf{0x_1 + 2x_2 + x_3 = 0 \ldots\ldots 4)}$

$-4x_1 + 4x_2 + 6x_3 = 0$

Divide by 2

$\therefore \mathbf{-2x_1 + 2x_2 + 3x_3 = 0 \ldots\ldots 5)}$

Solving 3) & 4) Using Crammers Rule,

$$\frac{x_1}{\begin{vmatrix} 0 & -1 \\ 2 & 1 \end{vmatrix}} = \frac{-x_2}{\begin{vmatrix} 1 & -1 \\ 0 & 1 \end{vmatrix}} = \frac{x_3}{\begin{vmatrix} 1 & 0 \\ 0 & 2 \end{vmatrix}}$$

$$\therefore \frac{x_1}{0-(-2)} = \frac{-x_2}{1-0} = \frac{x_3}{2-0}$$

$$\therefore \frac{x_1}{2} = \frac{-x_2}{1} = \frac{x_3}{2}$$

$$\therefore \frac{x_1}{2} = \frac{x_2}{-1} = \frac{x_3}{2}$$

$\therefore \lambda = -3$ **Corresponding Eigen vector is** $\begin{bmatrix} \mathbf{2} \\ \mathbf{-1} \\ \mathbf{2} \end{bmatrix}$

- **Homework:**

Find the Eigen Values & Eigen vectors of following matrix.

1. $A = \begin{bmatrix} 3 & 1 & 4 \\ 0 & 2 & 6 \\ 0 & 0 & 5 \end{bmatrix}$
2. $A = \begin{bmatrix} 2 & 1 & -1 \\ 1 & 1 & -2 \\ -1 & -2 & 1 \end{bmatrix}$
3. $A = \begin{bmatrix} 4 & 2 & -2 \\ -5 & 3 & 2 \\ -2 & 4 & 1 \end{bmatrix}$ **[S-24 4M]**
4. $A = \begin{bmatrix} 14 & -10 \\ 5 & -1 \end{bmatrix}$ **[S-23, W-23 4M]**