UNIT 3

VECTORS AND TENSORS (14 MARKS)

Topic Content: 3.1 Introduction, Definition of scalar and vector quantity, Representation of vector, Magnitude of vector, Component of vector, Direction ratio, Direction cosines

3.2 Types of vectors: Zero vector, Unit vector, Position vector, Equal vector, Negative vector. Parallel vector, Co-initial vector, Collinear vector

3.3 Algebra of vectors: Addition of vectors, Triangle law of vectors addition, Parallelogram law of Vectors addition, Subtraction of vectors, Multiplication of vectors by scalar

3.4 Product of two vectors: Scalar (dot) product of two vectors, Projection of one vector on another vector, Angle between two vectors using scalar(dot) product, Properties of scalar(dot) product

3.5 Vector (cross) product of two vectors, Angle between two vectors using vector(cross) product, Properties of vector (cross) product

3.6 Scalar triple product of vectors

3.7 Tensor: Definition of tensors, Types of tensors, Rank of tensors, Algebra of tensors

Course Outcome: After completion of this course, students will be able to

CO3: Build programs to implement basic operations based on vectors and tensors.

- ## INTRODUCTION:

Vectors, in Maths, are objects which have both, magnitude and direction. Magnitude defines the size of the vector. It is represented by a line with an arrow, where the length of the line is the magnitude of the vector and the arrow shows the direction. It is also known as **Euclidean vector** or **Geometric vector** or **spatial vector** or simply "**vector**".

- ## DEFINITIONS:

1. **Scalar:** Scalar is a quantity which has magnitude only with no direction.

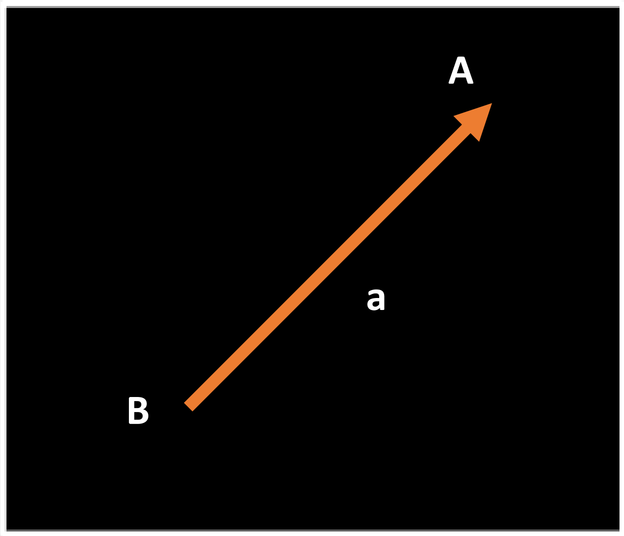
   Ex. Mass, Time, etc.

2. **Vector:** The vectors are defined as an object containing both magnitude and direction. Vector describes the movement of an object from one point to another.

   Ex. Velocity, Acceleration, Force, etc.

➢ REPRESENTATION OF VECTOR:

As we know already, a vector has both magnitude and direction. In the figure below, the length of the line AB is the magnitude and head of the arrow points towards the direction.



Therefore, vectors between two points A and B is given as $\overrightarrow{AB}$, or vector a. The arrow over the head of the vector shows the direction of the vector.

Vector in 2 dimensions: $\vec{v} = (2,3) = 2\vec{\imath} + 3\vec{\jmath}$

Vector in 3 dimensions: $\vec{v} = (-1,2,3) = -\vec{\imath} + 2\vec{\jmath} + 3\vec{k}$

➢ MAGNITUDE OF A VECTOR:

The magnitude of a vector $\vec{v} = a\vec{\imath} + b\vec{\jmath} + c\vec{k}$ is shown by vertical lines on both the sides of the given vector $|v|$. It represents the length of vector and calculated by following formula,

$\boxed{|\vec{v}| = \sqrt{a^2 + b^2 + c^2}}$

- **Example:**

1. Find the magnitude of a vector $(-1, 2, 4)$

Sol: Consider $\vec{v} = (-1,2,4) = -\vec{\imath} + 2\vec{\jmath} + 4\vec{k}$

Here, $1 = -1, b = 2, c = 4$

Using formula, $|\vec{v}| = \sqrt{a^2 + b^2 + c}$

$\therefore |\vec{v}| = \sqrt{(-1)^2 + (2)^2 + (4)^2}$

$\boxed{\therefore |\vec{v}| = \sqrt{21}}$

2. Find the magnitude of a vector $2\vec{\imath} - 4\vec{\jmath} + 4\vec{k}$

Sol: Consider $\vec{v} = 2\vec{\imath} - 4\vec{\jmath} + 4\vec{k}$

Here, $a = 2, b = -4, c = 4$

Using formula, $|\vec{v}| = \sqrt{a^2 + b^2 + c}$

$\therefore |\vec{v}| = \sqrt{(2)^2 + (4)^2 + (4)^2}$

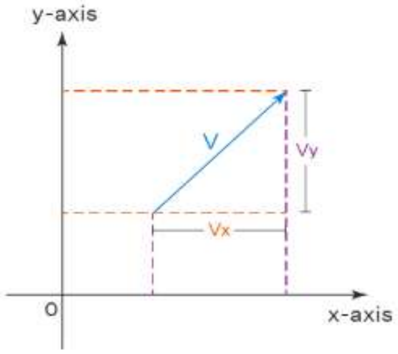
$\boxed{\therefore |\vec{v}| = \sqrt{36} = 6}$

- **Homework: Find the magnitude of a vector 3**$\vec{\imath} - 2\vec{\jmath} + \vec{k}$ **[Ans: $\sqrt{14}$]**

- **Component of a vector:**

- **In Two Dimension:**

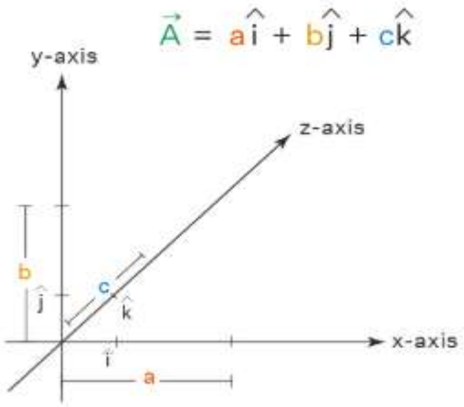
In a two-dimensional coordinate system, the direction of the vector is the angle made by the vector with the positive x-axis. Let V be the vector and θ is the angle made by the vector with the positive x-axis. Further, we have the components of this vector along the x and y axis as V_x, and V_y respectively. These components can be computed using the following expressions.



$\boldsymbol{V_x = V.cos\theta}$, **and** $\boldsymbol{V_y = V.sin\theta}$.

- **In Three Dimension:**

The vectors are also represented as $\vec{A} = a\vec{i} + b\vec{j} + c\vec{k}$in the three-dimensional space. Here $\vec{i}$, $\vec{j}$, $\vec{k}$ are the unit vectors along the x-axis, y-axis, and z-axis respectively. These unit vectors help in identifying the components of the vectors with reference to each of the axes. The components of vector A with respect to the x-axis, y-axis, z-axis, are a, b, c respectively.



- **Example:**

1. **Find the x and y components of a vector having a magnitude of 12 and making an angle of 45 degrees with the positive x-axis.**

   Sol: The given vector is V= 12, and it makes an angle θ = 45º.

   The x component of the vector is $V_x = V.cos\theta = 12.cos45^0 = 12.\frac{1}{\sqrt{2}} = 6\sqrt{2}$

   The y component of the vector is $V_y = V.sin\theta = 12.sin45^0 = 12.\frac{1}{\sqrt{2}} = 6\sqrt{2}$

   $\boxed{\therefore \boldsymbol{V_x = 6\sqrt{2}} \;\&\; \boldsymbol{V_y = 6\sqrt{2}}}$

2. **Find the vector from the components of a vector, having the x-component of 5 units, y-component of 12 units, and z-component of 4 units respectively.**

   Sol: Given, x component of vector is $a = 5$, y component of vector is $b = 12$, z component of vector is $c = 4$.

   The required vector is $\vec{V} = a\vec{i} + b\vec{j} + c\vec{k}$

   $\boxed{\therefore \boldsymbol{\vec{V} = 5\vec{i} + 12\vec{j} + 4\vec{k}}}$

- **Direction Ratios:**

  Direction ratios are the components of a vector along the x – axis, y – axis, & z – axis resp.

  The direction ratios of a vector $\vec{V} = a\vec{i} + b\vec{j} + c\vec{k}$ are $a,\ b,\ c$.

- **Direction Cosines:**

  The direction cosines of a vector $\vec{V} = a\vec{i} + b\vec{j} + c\vec{k}$ are given by,

  $$\boxed{l = cos\alpha = \frac{a}{|\vec{V}|}, m = cos\beta = \frac{b}{|\vec{V}|}, n = cos\gamma = \frac{c}{|\vec{V}|}.}$$

- **Example:**

1. **Find direction ratios & direction cosines of a vector $(1, 2, -3)$**

   Sol: Consider, $\vec{V} = (1,2,-3) = \vec{i} + 2\vec{j} - 3\vec{k}$

   Here, $a = 1, b = 2, c = -3$

   Direction ratios are $\boxed{a = 1, b = 2, c = -3}$

   Now, $|\vec{V}| = \sqrt{(1)^2 + (2)^2 + (-3)^2}$

   $\boxed{\therefore |\vec{V}| = \sqrt{14}}$

   Direction Cosines are,

   $l = \frac{a}{|\vec{V}|} = \frac{1}{\sqrt{14}}$

   $m = \frac{b}{|\vec{V}|} = \frac{2}{\sqrt{14}}$

   $n = \frac{c}{|\vec{V}|} = \frac{-3}{\sqrt{14}}$

   $\boxed{\therefore D.C. = \left(\frac{1}{\sqrt{14}}, \frac{2}{\sqrt{14}}, \frac{-3}{\sqrt{14}}\right)}$

2. **Find direction ratios & direction cosines of a vector $\vec{A} = 2\vec{i} + \vec{j} + \vec{k}$**

   Sol: Given, $\vec{A} = 2\vec{i} + \vec{j} + \vec{k}$

   Here, $a = 2, b = 1, c = 1$

   Direction ratios are $\boxed{a = 2, b = 1, c = 1}$

   Now, $|\vec{V}| = \sqrt{(2)^2 + (1)^2 + (1)^2}$

   $\boxed{\therefore |\vec{V}| = \sqrt{6}}$

   Direction Cosines are,

   $l = \frac{a}{|\vec{V}|} = \frac{2}{\sqrt{6}}$

   $m = \frac{b}{|\vec{V}|} = \frac{1}{\sqrt{6}}$

   $n = \frac{c}{|\vec{V}|} = \frac{1}{\sqrt{6}}$

   $\boxed{\therefore D.C. = \left(\frac{2}{\sqrt{6}}, \frac{1}{\sqrt{6}}, \frac{1}{\sqrt{6}}\right)}$

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Course Outcome: After completion of this course, students will be able to

CO3: Build programs to implement basic operations based on vectors and tensors.

- **Types of Vectors:**

1. Zero or Null Vector:

A vector with magnitude zero is called as zero or null vector and denoted by $\vec{0}$.

2. Unit Vector:

A vector with magnitude one is called as unit vector and denoted by $\vec{u}$.

Unit vector of any vector $\vec{a}$ is given by $\boxed{\vec{u} = \frac{\vec{a}}{|\vec{a}|}}$

- **Example:**

1. Find unit vector of a vector $\vec{a} = \vec{\imath} - 2\vec{\jmath} + 3\vec{k}$

Sol: Given $\vec{a} = \vec{\imath} - 2\vec{\jmath} + 3\vec{k}$

Here, $a = 1, b = -2, c = 3$

Using Formula, $\boxed{\vec{u} = \frac{\vec{a}}{|\vec{a}|}}$ ……. (1)

$\therefore |\vec{a}| = \sqrt{a^2 + b^2 + c^2}$

$\therefore |\vec{a}| = \sqrt{(1)^2 + (-2)^2 + (3)^2} \rightarrow \boxed{|\vec{a}| = \sqrt{14}}$

Equation 1) becomes;

$$\boxed{\vec{u} = \frac{\vec{\imath} - 2\vec{\jmath} + 3\vec{k}}{\sqrt{14}}}$$

3. **Position Vector:**

A position vector is defined as a vector that symbolises either the position or the location of any given point with respect to any arbitrary reference point like the origin. The direction of the position vector always points from the origin of that vector towards a given point.

Consider two points A and B whose coordinates are (x_1, y_1, z_1) and (x_2, y_2, z_1), respectively. To determine the position vector, we need to subtract the corresponding components of A from B as follows:

$\therefore \overrightarrow{AB} = \vec{b} - \vec{a}$

$\boxed{\therefore \overrightarrow{AB} = (x_2 - x_1)\vec{i} + (y_2 - y_1)\vec{j} + (z_2 - z_1)\vec{k}}$

4. **Equal Vectors:**

Two or more vectors are said to be equal when their magnitude is equal and also their direction is the same.

Two vectors $\vec{a} = a_1\vec{i} + b_1\vec{j} + c_1\vec{k}$ & $\vec{b} = a_2\vec{i} + b_2\vec{j} + c_2\vec{k}$ are said to be equal iff

$$\boxed{a_1 = a_2, b_1 = b_2 \ \& \ c_1 = c_2}$$

- **Example:**

1. **Find the value of P if the vector $\vec{a} = P\vec{i} + 5\vec{j} + \vec{k}$ & $\vec{b} = 2\vec{i} - \vec{j} + 3\vec{k}$ are equal.** **[W-23 2M]**

Sol: Given $\vec{a} = P\vec{i} + 5\vec{j} + \vec{k}$ & $\vec{b} = 2\vec{i} - \vec{j} + 3\vec{k}$ are equal.

By definition of equality of vectors.

Two vectors $\vec{a} = a_1\vec{i} + b_1\vec{j} + c_1\vec{k}$ & $\vec{b} = a_2\vec{i} + b_2\vec{j} + c_2\vec{k}$ are said to be equal iff

$$\boxed{a_1 = a_2, b_1 = b_2 \ \& \ c_1 = c_2}$$

$\boxed{\therefore P = 2}$

5. **Negative Vector:**

If two vectors are the same in magnitude but exactly opposite in direction then both the vectors are negative of each other. Assume there are two vectors **a** and **b,** such that these vectors are exactly the same in magnitude but opposite in direction then these vectors can be given by

$\boxed{a = -b}$

6. **Parallel Vector: (Collinear Vector)**

Two vectors $\vec{a}$ & $\vec{b}$ are said to be parallel iff one vector is scalar multiple of the other vector.

i.e. $\boxed{\vec{a} = k\vec{b}}$

$\vec{a} = 2\vec{i} - 3\vec{j} + 6\vec{k}$ & $\vec{b} = 4\vec{i} - 6\vec{j} + 12\vec{k}$

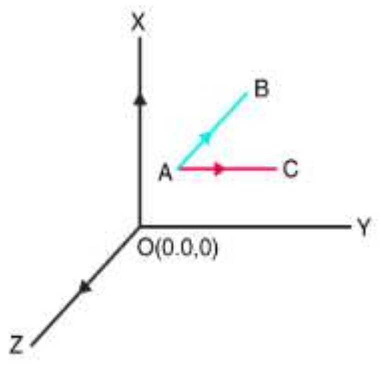
$\therefore \vec{b} = 4\vec{i} - 6\vec{j} + 12\vec{k}$

$\therefore \vec{b} = 2(2\vec{i} - 3\vec{j} + 6\vec{k})$

$\boxed{\therefore \vec{b} = 2\,\vec{a}}$

7. **Co-initial Vectors.**

The vectors which have the same starting point are called co-initial vectors.

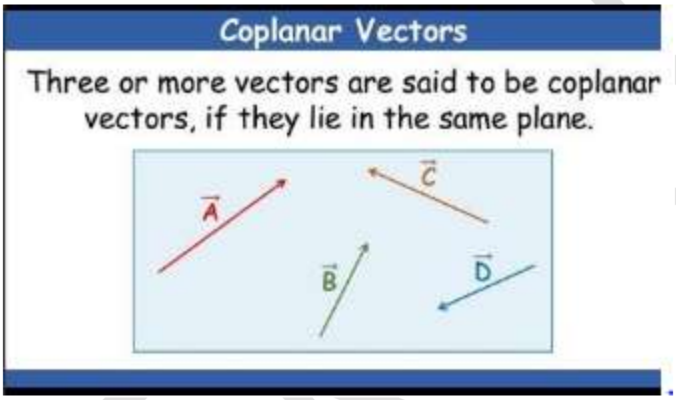


The vectors $\overrightarrow{AB}$ & $\overrightarrow{AC}$ are called co-initial vectors as they have the same starting point.

Note: **Two vectors are said to be collinear iff their cross product is equal to zero.**

8. **Coplanar vectors.**

Three or more vectors lying in the same plane or parallel to the same plane are known as co-planar vectors.



Note: **Three vectors are said to be coplanar iff their scalar triple product is equal to zero.**

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- **Algebra of Vectors:**

  - **Addition of Vectors:**

The vector addition is the sum of multiple (two or more) vectors.

If $\vec{a} = 2\vec{i} - 3\vec{j} + 4\vec{k}, \vec{b} = \vec{i} + 2\vec{j} - 5\vec{k}$ **find** $\vec{a} + \vec{b}$

Sol: $\vec{a} + \vec{b} = 2\vec{i} - 3\vec{j} + 4\vec{k} + \vec{i} + 2\vec{j} - 5\vec{k}$

$\therefore \vec{a} + \vec{b} = (2+1)\vec{i} + (-3+2)\vec{j} + (4-5)\vec{k}$

$\boxed{\therefore \vec{a} + \vec{b} = 3\vec{i} - \vec{j} - \vec{k}}$

  - **Subtraction of Vectors:**

If $\vec{a} = 2\vec{i} - 3\vec{j} + 4\vec{k}, \vec{b} = \vec{i} + 2\vec{j} - 5\vec{k}$ **find** $\vec{a} - \vec{b}$

Sol: $\vec{a} - \vec{b} = 2\vec{i} - 3\vec{j} + 4\vec{k} - (\vec{i} + 2\vec{j} - 5\vec{k})$

$\therefore \vec{a} - \vec{b} = (2-1)\vec{i} + (-3-2)\vec{j} + (4-(-5))\vec{k}$

$\boxed{\therefore \vec{a} - \vec{b} = \vec{i} - 5\vec{j} + 9\vec{k}}$

  - **Multiplication of Vector by Scalar.**

If $\vec{a} = 2\vec{i} - 3\vec{j} + 4\vec{k}$ **find** $3\vec{a}$

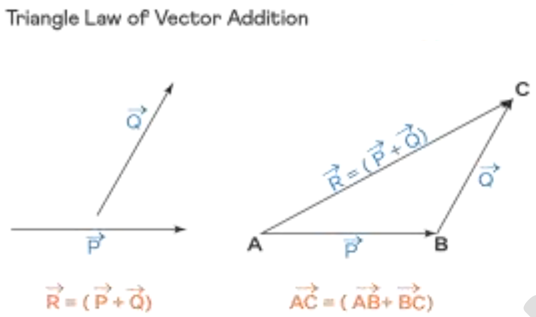
Sol: $3\vec{a} = 3(2\vec{i} - 3\vec{j} + 4\vec{k})$

$\boxed{\therefore 3\vec{a} = 6\vec{i} - 9\vec{j} + 12\vec{k}}$

Triangle Law of Vector Addition

Triangle Law of Vector Addition is a mathematical concept that is used to find the sum of two vectors. This law is used to add two vectors when the first vector's head is joined to the tail of the second vector and then joining the tail of the first vector to the head of the second vector to form a triangle, and hence obtain the resultant sum vector. That's why the triangle law of vector addition is also called the head-to-tail method for the addition of vectors.

Let us study the triangle law of the addition of vectors, its statement, and formula. This law is used to determine the net displacement, velocity, acceleration, etc. We will also solve questions and examples based on the triangle law of vector addition to understand its application and the concept.



Triangle Law of Vector Addition Formula

Consider two vectors P and Q such that the angle between them is θ and their resultant sum vector using the triangle law of vector addition is given by the vector R. The formula for the magnitude $|R|$ and direction ϕ of the resultant vector R using triangle law for the addition of vectors is given by,

$$\boxed{|R| = \sqrt{P^2 + 2PQ\cos\theta + Q^2}}$$

$$\boxed{\phi = \tan^{-1}\left[\frac{Q\sin\theta}{P + Q\cos\theta}\right]}$$

Important Notes on Triangle Law of Vector Addition

- Triangle law of vector addition is used to find the sum of two vectors when the head of the first vector is joined to the tail of the second vector.
- Magnitude of the resultant sum vector R: $|R| = \sqrt{P^2 + 2PQ\cos\theta + Q^2}$
- Direction of the resultant vector R: $\phi = \tan^{-1}\left[\frac{Q\sin\theta}{P + Q\cos\theta}\right]$

- **Example 1: Given two vectors, $\vec{P} = (2, 5)$ and $\vec{Q} = (3, -2)$, determine the magnitude of the resultant vector $\vec{R}$ using their components.**

  Solution: Given $\vec{P} = (2, 5) = 2\vec{i} + 5\vec{j}$ & $\vec{Q} = (3, -2) = 3\vec{i} - 2\vec{j}$

  Resultant vector $\vec{R}$ is given by triangle law:

  $\vec{R} = \vec{P} + \vec{Q}$

  $\therefore \vec{R} = 2\vec{i} + 5\vec{j} + 3\vec{i} - 2\vec{j}$

  $\boxed{\therefore \vec{R} = 5\vec{i} + 3\vec{j}}$

  Now, $|\vec{R}| = \sqrt{(5)^2 + (3)^2}$

  $\boxed{\therefore |\vec{R}| = \sqrt{34}}$

- **Example 2: Two vectors A and B have magnitudes of 4 units and 9 units and make an angle of 30° with each other. Find the magnitude and direction of the resultant sum vector using the triangle law of vector addition formula.**

  Solution: The formula for the resultant vector using the triangle law are:

  $|R| = \sqrt{A^2 + 2AB\,cos\theta + B^2}$

  $\phi = \tan^{-1}\left[\dfrac{B\,sin\,\theta}{A + B\,cos\,\theta}\right]$

  So, we have

  $|R| = \sqrt{A^2 + 2AB\,cos\theta + B^2})$

  $\therefore |R| = \sqrt{4^2 + 2 \times 4 \times 9\,cos30° + 9^2}$

  $\therefore |R| = \sqrt{16 + 72 \times \dfrac{\sqrt{3}}{2} + 81}$

  $\therefore |R| = \sqrt{97 + 36\sqrt{3}}$

  $\boxed{\therefore |R| = 12.623}$ Units

  The direction of R is given by,

  $\phi = \tan^{-1}\left[\dfrac{B\,sin\,\theta}{A + B\,cos\,\theta}\right]$

  $\therefore \phi = \tan^{-1}\left[\dfrac{9\,sin\,30^0}{4 + 9\,cos\,30^0}\right]$

  $\therefore \phi = \tan^{-1}\left[\dfrac{9 \times \frac{1}{2}}{4 + 9 \times \frac{\sqrt{3}}{2}}\right]$

  $\boxed{\therefore \phi = 20.87°}$ **Answer:** Hence, the magnitude of the resultant vector is 12.623 units and the direction is 20.87°, approximately.

- **Example 3: Two vectors with magnitudes 2 units and $\sqrt{2}$ units act on a body. The resultant vector has a magnitude of $\sqrt{10}$ units. Find the angle between the two given vectors.**

  Solution: Assume the two given vectors to be P and Q such that |P| = 2 and |Q| = $\sqrt{2}$.

  Then, using the formula for the triangle law of vector addition, we have

  $|R| = \sqrt{P^2 + 2PQ\ cos\theta + Q^2}$

  $\therefore \sqrt{10} = \sqrt{2^2 + 2 \times 2 \times \sqrt{2}\ cos\theta + \left(\sqrt{2}\right)^2}$

  $\therefore \sqrt{10} = \sqrt{4 + 4\sqrt{2}\ cos\theta + 2}$

  Squaring on both sides.

  $\therefore 10 = 4 + 4\sqrt{2}\ cos\theta + 2$

  $\therefore 10 = 6 + 4\sqrt{2}\ cos\theta$

  $\therefore 10 - 6 = 4\sqrt{2}\ cos\theta$

  $\therefore \frac{4}{4\sqrt{2}} = cos\theta \rightarrow cos\theta = \frac{1}{\sqrt{2}} \rightarrow \theta = \cos^{-1}\left(\frac{1}{\sqrt{2}}\right)$

  $\Rightarrow \boxed{\boldsymbol{\theta = 45°}}$

  Answer: Hence the angle between the two vectors is 45°.

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- **Product of two Vectors:**

- **Scalar (Dot) product of two Vectors:**

The scalar or dot product of two vectors $\vec{a}$ and $\vec{b}$ of magnitude $|\vec{a}|$ and $|\vec{b}|$ is given as $\boxed{\vec{a}.\vec{b} = |\vec{a}||\vec{b}|\, cos\, \theta}$,

where θ represents the angle between the vectors $\vec{a}$ and $\vec{b}$ taken in the direction of the vectors.

- **Dot Product Properties of Vector:**

1. Dot product of two vectors is commutative i.e. $\vec{a}.\vec{b} = \vec{b}.\vec{a} = \boldsymbol{ab\, cos\, \theta}$.
2. If $\vec{a}.\vec{b} = 0$ then Two vectors are perpendicular,

   it can be clearly seen that either b or a is zero or $cos\,\theta = 0 \to \boldsymbol{\theta} = \frac{\pi}{2}$.
3. The dot product of a vector to itself is the magnitude squared of the vector i.e.

   $\vec{a}.\vec{a} = a.a\ cos\, 0 = a^2$
4. The dot product follows the distributive law also i.e. $a.(b + c) = a.b + a.c$
5. In terms of orthogonal coordinates for mutually perpendicular vectors it is seen that

   $\vec{i}.\vec{i} = \vec{j}.\vec{j} = \vec{k}.\vec{k} = 1$
6. In terms of unit vectors, if $\vec{a} = a_1\vec{i} + b_1\vec{j} + c_1\vec{k}$ & $\vec{b} = a_2\vec{i} + b_2\vec{j} + c_2\vec{k}$

   Then $\vec{a}.\vec{b} = (a_1\vec{i} + b_1\vec{j} + c_1\vec{k}).(a_2\vec{i} + b_2\vec{j} + c_2\vec{k})$

   $\boxed{\therefore \vec{a}.\vec{b} = a_1a_2 + b_1b_2 + c_1.c_2}$

- **Examples:**

1. **Find $\bar{a}.\bar{b}$ if $\bar{a} = 2\hat{\imath} - 3\hat{\jmath} + \hat{k}$ & $\bar{b} = \hat{\imath} + 2\hat{\jmath} - 3\hat{k}$ [S-24 2M]**

   Sol: Consider,

   $\bar{a}.\bar{b} = (2\hat{\imath} - 3\hat{\jmath} + \hat{k}).(\hat{\imath} + 2\hat{\jmath} - 3\hat{k})$

   $\therefore \bar{a}.\bar{b} = 2.1 + (-3).2 + 1.(-3)$

   $\therefore \bar{a}.\bar{b} = 2 - 6 - 3$

   $\boxed{\therefore \bar{a}.\bar{b} = -7}$

2. **Find the value of p if the vectors $\bar{a} = p\bar{\imath} + 5\bar{\jmath} + \bar{k}$ & $\bar{b} = 2\bar{\imath} - \bar{\jmath} - 5\bar{k}$ are perpendicular to each other. [S-23 2M]**

   Sol: We know that, two vectors are perpendicular to each other iff their dot product is equal to zero.

   $\therefore \bar{a}.\bar{b} = 0$

   $\therefore (p\bar{\imath} + 5\bar{\jmath} + \bar{k}).(2\bar{\imath} - \bar{\jmath} - 5\bar{k}) = 0$

   $\therefore p.2 + 5.(-1) + 1.(-5) = 0$

   $\therefore 2p - 5 - 5 = 0$

   $\therefore 2p - 10 = 0$

   $\therefore 2p = 10$

   $\therefore p = \frac{10}{2}$

   $\boxed{\therefore p = 5}$

3. **Let there be two vectors |a|=4 and |b|=2 and θ = 60°. Find their dot product.**

   Sol: We know that, $\vec{a}.\vec{b} = |\vec{a}||\vec{b}|\cos\theta$

   $\therefore \vec{a}.\vec{b} = 4.2\cos 60^0$

   $\therefore \vec{a}.\vec{b} = 8 \times \frac{1}{2}$

   $\boxed{\therefore \vec{a}.\vec{b} = 4}$

- **Homework:**

1. **Find $\bar{a}.\bar{b}$ if $\bar{a} = 2\hat{\imath} - 3\hat{\jmath} + 4\hat{k}$ & $\bar{b} = \hat{\imath} + 2\hat{\jmath} - 5\hat{k}$** [Ans: -24]
2. **Find $\bar{a}.\bar{b}$ if $\bar{a} = 2\hat{\imath} + 5\hat{\jmath} + \hat{k}$ & $\bar{b} = 2\hat{\imath} - \hat{\jmath} + 3\hat{k}$** [Ans: -2]
3. **Let there be two vectors $(6, 2, -1)$ & $(5, -8, 2)$. Find the dot product of the vectors.** [Ans: 12]
4. **Find the dot product of two vectors having magnitudes of 6 units and 7 units, and the angle between the vectors is 60°.** [Ans: 21]

- **Projection of vector:**

Projection of a vector $\bar{a}$ onto the vector $\bar{b}$ is denoted by $proj._{\bar{b}}\ \bar{a}$ & is defined as, $\boxed{\boldsymbol{proj._{\bar{b}}\ \bar{a} = \frac{\bar{a}.\bar{b}}{|\bar{b}|}}}$

❖ **Examples:**

1. **Find the projection of $\bar{a} = 2\bar{\imath} + \bar{\jmath} + \bar{k}$ on $\bar{b} = \bar{\imath} + 3\bar{\jmath} + \bar{k}$ [S-23 W-23 2M]**

Sol: We know that,

$$\boldsymbol{proj._{\bar{b}}\ \bar{a} = \frac{\bar{a}.\bar{b}}{|\bar{b}|}} \ldots\ldots..1)$$

$\therefore \bar{a}.\bar{b} = (2\bar{\imath} + \bar{\jmath} + \bar{k}).(\bar{\imath} + 3\bar{\jmath} + \bar{k})$

$\therefore \bar{a}.\bar{b} = 2.1 + 1.3 + 1.1$

$\boxed{\therefore \bar{a}.\bar{b} = 6}$

Now, $|\bar{b}| = \sqrt{(1)^2 + (3)^2 + (1)^2}$

$\boxed{\therefore |\bar{b}| = \sqrt{11}}$

Equation 1) becomes,

$$\boxed{\therefore proj._{\bar{b}}\ \bar{a} = \frac{6}{\sqrt{11}}}$$

2. **Find the projection of $\bar{a} = 4\bar{\imath} + 2\bar{\jmath} + \bar{k}$ on $\bar{b} = 5\bar{\imath} - 3\bar{\jmath} + 3\bar{k}$**

Sol: We know that,

$$\boldsymbol{proj._{\bar{b}}\ \bar{a} = \frac{\bar{a}.\bar{b}}{|\bar{b}|}} \ldots\ldots..1)$$

$\therefore \bar{a}.\bar{b} = (4\bar{\imath} + 2\bar{\jmath} + \bar{k}).(5\bar{\imath} - 3\bar{\jmath} + 3\bar{k})$

$\therefore \bar{a}.\bar{b} = 4.5 + 2.(-3) + 1.3$

$\boxed{\therefore \bar{a}.\bar{b} = 17}$

Now, $|\bar{b}| = \sqrt{(5)^2 + (-3)^2 + (3)^2}$

$\boxed{\therefore |\bar{b}| = \sqrt{43}}$

Equation 1) becomes,

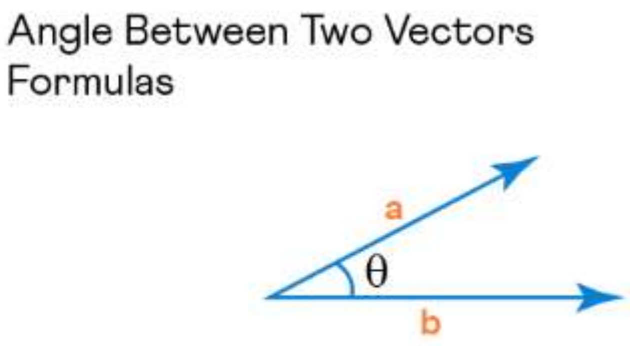
$$\boxed{\therefore proj._{\bar{b}}\ \bar{a} = \frac{17}{\sqrt{43}}}$$

❖ **Homework:**

1. **Find the projection of $\bar{a} = 2\bar{\imath} + 3\bar{\jmath} + 2\bar{k}$ on $\bar{b} = \bar{\imath} + 2\bar{\jmath} + \bar{k}$** $\left[Ans: \frac{10}{\sqrt{6}}\right]$
2. **Find the projection of $\bar{a} = 5\bar{\imath} + 4\bar{\jmath} + \bar{k}$ on $\bar{b} = 3\bar{\imath} + 5\bar{\jmath} - 2\bar{k}$** $\left[Ans: \frac{33}{\sqrt{38}}\right]$
3. **Find the projection of vector of 12 units, on another vector, and the angle between the two vectors is 60 degrees.** $[Ans: 6]$

- **Angle between two vectors (Cosine of the angle):**

Cosine angle between two vectors $\bar{a}$ and $\bar{b}$ is given by formula,



$$\boxed{\boldsymbol{\theta = \cos^{-1}\left[\frac{\bar{a}.\bar{b}}{|\bar{a}||\bar{b}|}\right]}}$$

❖ **Example:**

1. Compute the angle between two vectors 3i + 4j – k and 2i – j + k. [S-24 4M]

Sol: Consider, $\bar{a} = 3\bar{\imath} + 4\bar{\jmath} - \bar{k}$ & $\bar{b} = 2\bar{\imath} - \bar{\jmath} + \bar{k}$

Angle between two vectors is given by,

$$\boldsymbol{\theta = \cos^{-1}\left[\frac{\bar{a}.\bar{b}}{|\bar{a}||\bar{b}|}\right] \ldots\ldots\ldots 1)}$$

$$\therefore \bar{a}.\bar{b} = (3\bar{\imath} + 4\bar{\jmath} - \bar{k}).(2\bar{\imath} - \bar{\jmath} + \bar{k})$$

$$\therefore \bar{a}.\bar{b} = 3.2 + 4.(-1) + (-1).1 = 6 - 4 - 1$$

$$\boxed{\boldsymbol{\therefore \bar{a}.\bar{b} = 1}}$$

Now, $|\bar{a}| = \sqrt{(3)^2 + (4)^2 + (-1)^2}$

$$\boxed{\boldsymbol{\therefore |\bar{a}| = \sqrt{26}}}$$

$$|\bar{b}| = \sqrt{(2)^2 + (-1)^2 + (1)^2}$$

$$\boxed{\boldsymbol{\therefore |\bar{b}| = \sqrt{6}}}$$

Equation 1) becomes,

$$\theta = \cos^{-1}\left[\frac{1}{\sqrt{26}\,\sqrt{6}}\right]$$

$$\therefore \boxed{\boldsymbol{\theta = \cos^{-1}\left[\frac{1}{\sqrt{156}}\right]}} \quad Or \quad \boxed{\boldsymbol{\theta = 85.4078^0}}$$

2. Find the angle between the vectors $\bar{a} = 2\bar{\imath} + 2\bar{\jmath} + \bar{k}$ & $\bar{b} = 3\bar{\imath} + 6\bar{\jmath} + 2\bar{k}$ [S-23 W-23 4M]

Sol: Consider, $\bar{a} = 2\bar{\imath} + 2\bar{\jmath} + \bar{k}$ & $\bar{b} = 3\bar{\imath} + 6\bar{\jmath} + 2\bar{k}$

Angle between two vectors is given by,

$$\boldsymbol{\theta = \cos^{-1}\left[\frac{\bar{a}.\bar{b}}{|\bar{a}||\bar{b}|}\right] \ldots\ldots\ldots 1)}$$

$\therefore \bar{a}.\bar{b} = (2\bar{\imath} + 2\bar{\jmath} + \bar{k}).(3\bar{\imath} + 6\bar{\jmath} + 2\bar{k})$

$\therefore \bar{a}.\bar{b} = 2.3 + 2.6 + 1.2 = 6 + 12 + 2$

$\boxed{\therefore \bar{a}.\bar{b} = 20}$

Now, $|\bar{a}| = \sqrt{(2)^2 + (2)^2 + (1)^2}$

$\boxed{\therefore |\bar{a}| = 3}$

$|\bar{b}| = \sqrt{(3)^2 + (6)^2 + (2)^2}$

$\boxed{\therefore |\bar{b}| = 7}$

Equation 1) becomes,

$\therefore \theta = \cos^{-1}\left[\frac{20}{21}\right]$

$\therefore \boxed{\theta = 17.7528^0}$

3. **Find the angle between the vectors $\bar{a} = 2\bar{\imath} - \bar{\jmath} + 3\bar{k}$ & $\bar{b} = 2\bar{\imath} + \bar{k}$**

Sol: Consider, $\bar{a} = 2\bar{\imath} - \bar{\jmath} + 3\bar{k}$ & $\bar{b} = 2\bar{\imath} + 0\bar{\jmath} + \bar{k}$

Angle between two vectors is given by,

$$\boldsymbol{\theta = \cos^{-1}\left[\frac{\bar{a}.\bar{b}}{|\bar{a}||\bar{b}|}\right] \ldots\ldots\ldots 1)}$$

$\therefore \bar{a}.\bar{b} = (2\bar{\imath} - \bar{\jmath} + 3\bar{k}).(2\bar{\imath} + 0\bar{\jmath} + \bar{k})$

$\therefore \bar{a}.\bar{b} = 2.2 + (-1).0 + 3.1 = 4 + 0 + 3$

$\boxed{\therefore \bar{a}.\bar{b} = 7}$

Now, $|\bar{a}| = \sqrt{(2)^2 + (-1)^2 + (3)^2}$

$\boxed{\therefore |\bar{a}| = \sqrt{14}}$

$|\bar{b}| = \sqrt{(2)^2 + (0)^2 + (1)^2}$

$\boxed{\therefore |\bar{b}| = \sqrt{5}}$

Equation 1) becomes,

$\therefore \theta = \cos^{-1}\left[\frac{7}{\sqrt{14}\sqrt{5}}\right]$

$\therefore \boxed{\theta = 33.2109^0}$

❖ **Homework:**

1. **Find the angle between the vectors $\bar{a} = -4\bar{\imath} + 2\bar{\jmath} + \bar{k}$ & $\bar{b} = 4\bar{\imath} + 3\bar{k}$** $[Ans: 124.5668^0]$
2. **Find the angle between the vectors $\bar{a} = 2\bar{\imath} + 3\bar{\jmath} - \bar{k}$ & $\bar{b} = \bar{\imath} - 3\bar{\jmath} + 5\bar{k}$** $[Ans: 122.82^0]$

UNIT 3

VECTORS AND TENSORS (14 MARKS)

Topic Content: 3.1 Introduction, Definition of scalar and vector quantity, Representation of vector, Magnitude of vector, Component of vector, Direction ratio, Direction cosines

3.2 Types of vectors: Zero vector, Unit vector, Position vector, Equal vector, Negative vector. Parallel vector, Co-initial vector, collinear vector

3.3 Algebra of vectors: Addition of vectors, Triangle law of vectors addition, Parallelogram law of Vectors addition, Subtraction of vectors, Multiplication of vectors by scalar

3.4 Product of two vectors: Scalar (dot) product of two vectors, Projection of one vector on another vector, Angle between two vectors using scalar(dot) product, Properties of scalar(dot) product

3.5 Vector (cross) product of two vectors, Angle between two vectors using vector (cross) product, Properties of vector (cross) product

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3.6 Scalar triple product of vectors

3.7 Tensor: Definition of tensors, Types of tensors, Rank of tensors, Algebra of tensors

Course Outcome: After completion of this course, students will be able to

CO3: Build programs to implement basic operations based on vectors and tensors.

- **Product of two Vectors:**

- **Vector (Cross) product of two Vectors:**

The vector or cross product of two vectors $\vec{a}$ and $\vec{b}$ of magnitude $|\vec{a}|$ and $|\vec{b}|$ is given as

$\boxed{\vec{a} \times \vec{b} = |\vec{a}||\vec{b}| \sin\theta\, \hat{n}}$,

where θ represents the angle between the vectors $\vec{a}$ and $\vec{b}$ taken in the direction of the vectors & $\hat{n}$ is the unit vector perpendicular to both vectors $\vec{a}$ and $\vec{b}$.

Or in determinant form

In terms of unit vectors, if $\vec{a} = a_1\vec{\imath} + b_1\vec{\jmath} + c_1\vec{k}$ & $\vec{b} = a_2\vec{\imath} + b_2\vec{\jmath} + c_2\vec{k}$

$$\boxed{\vec{a} \times \vec{b} = \begin{vmatrix} \vec{\imath} & \vec{\jmath} & \vec{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}}$$

- **Cross Product Properties of Vector:**

1. $\vec{a} \times \vec{b} = -\vec{b} \times \vec{a}$
2. $\vec{a} \times (\vec{b} + \vec{c}) = \vec{a} \times \vec{b} + \vec{a} \times \vec{c}$
3. Sine angle between two vectors $\vec{a}$ and $\vec{b}$ is given by formula,

$$\boxed{\theta = \sin^{-1}\left[\frac{|\vec{a} \times \vec{b}|}{|\vec{a}|\,|\vec{b}|}\right]}$$

- **Examples:**

4. Find $\bar{a} \times \bar{b}$ if $\bar{a} = 2\hat{\imath} + \hat{\jmath} - 3\hat{k}$ & $\bar{b} = \hat{\imath} - 2\hat{\jmath} + \hat{k}$

Sol: Consider,

$$\bar{a} \times \bar{b} = \begin{vmatrix} \bar{\imath} & \bar{\jmath} & \bar{k} \\ 2 & 1 & -3 \\ 1 & -2 & 1 \end{vmatrix}$$

$$\therefore \bar{a} \times \bar{b} = \bar{\imath}\begin{vmatrix} 1 & -3 \\ -2 & 1 \end{vmatrix} - \bar{\jmath}\begin{vmatrix} 2 & -3 \\ 1 & 1 \end{vmatrix} + \bar{k}\begin{vmatrix} 2 & 1 \\ 1 & -2 \end{vmatrix}$$

$$\therefore \bar{a} \times \bar{b} = \bar{\imath}[1-6] - \bar{\jmath}[2+3] + \bar{k}[-4-1]$$

$$\boxed{\therefore \bar{a} \times \bar{b} = -5\bar{\imath} - 5\bar{\jmath} - 5\bar{k}}$$

5. If $\bar{a} = \bar{\imath} - 2\bar{\jmath} + 3\bar{k}, \bar{b} = 2\bar{\imath} + \bar{\jmath} - \bar{k}$ & $\bar{c} = \bar{\jmath} + \bar{k}$ find vector $\bar{a} \times (\bar{b} \times \bar{c})$ [W-23 4M]

Sol: Given,

$\bar{a} = \bar{\imath} - 2\bar{\jmath} + 3\bar{k}$,

$\bar{b} = 2\bar{\imath} + \bar{\jmath} - \bar{k}$ &

$\bar{c} = 0\bar{\imath} + \bar{\jmath} + \bar{k}$

$$\therefore \bar{b} \times \bar{c} = \begin{vmatrix} \bar{\imath} & \bar{\jmath} & \bar{k} \\ 2 & 1 & -1 \\ 0 & 1 & 1 \end{vmatrix}$$

$$\therefore \bar{b} \times \bar{c} = \bar{\imath}\begin{vmatrix} 1 & -1 \\ 1 & 1 \end{vmatrix} - \bar{\jmath}\begin{vmatrix} 2 & -1 \\ 0 & 1 \end{vmatrix} + \bar{k}\begin{vmatrix} 2 & 1 \\ 0 & 1 \end{vmatrix}$$

$$\therefore \bar{b} \times \bar{c} = \bar{\imath}[1+1] - \bar{\jmath}[2-0] + \bar{k}[2-0]$$

$$\boxed{\therefore \bar{b} \times \bar{c} = 2\bar{\imath} - 2\bar{\jmath} + 2\bar{k}}$$

Now,

$$\bar{a} \times (\bar{b} \times \bar{c}) = \begin{vmatrix} \bar{\imath} & \bar{\jmath} & \bar{k} \\ 1 & -2 & 3 \\ 2 & -2 & 2 \end{vmatrix}$$

$$\therefore \bar{a} \times (\bar{b} \times \bar{c}) = \bar{\imath}\begin{vmatrix} -2 & 3 \\ -2 & 2 \end{vmatrix} - \bar{\jmath}\begin{vmatrix} 1 & 3 \\ 2 & 2 \end{vmatrix} + \bar{k}\begin{vmatrix} 1 & -2 \\ 2 & -2 \end{vmatrix}$$

$$\therefore \bar{a} \times (\bar{b} \times \bar{c}) = \bar{\imath}[-4+6] - \bar{\jmath}[2-6] + \bar{k}[-2+4]$$

$$\boxed{\therefore \bar{a} \times (\bar{b} \times \bar{c}) = 2\bar{\imath} + 4\bar{\jmath} + 2\bar{k}}$$

6. If $\bar{a} = \bar{\imath} - \bar{\jmath}, \bar{b} = 2\bar{\jmath} + \bar{k}$ & $\bar{c} = 2\bar{\imath} - 3\bar{k}$ find vector $\bar{a} \times (\bar{b} + \bar{c}) = (\bar{a} \times \bar{b}) + (\bar{a} \times \bar{c})$ [S-24 4M]

Sol: Given,

$\bar{a} = \bar{\imath} - \bar{\jmath} + 0\bar{k}$,

$\bar{b} = 0\bar{\imath} + 2\bar{\jmath} + \bar{k}$ &

$\bar{c} = 2\bar{\imath} + 0\bar{\jmath} - 3\bar{k}$

$$\therefore \bar{b} + \bar{c} = 0\bar{\imath} + 2\bar{\jmath} + \bar{k} + 2\bar{\imath} + 0\bar{\jmath} - 3\bar{k}$$

$$\boxed{\therefore \bar{b} + \bar{c} = 2\bar{\imath} + 2\bar{\jmath} - 2\bar{k}}$$

Consider,

$L.H.S. = \bar{a} \times (\bar{b} + \bar{c})$

$\therefore L.H.S. = \begin{vmatrix} \bar{\imath} & \bar{\jmath} & \bar{k} \\ 1 & -1 & 0 \\ 2 & 2 & -2 \end{vmatrix}$

$\therefore L.H.S. = \bar{\imath}\begin{vmatrix} -1 & 0 \\ 2 & -2 \end{vmatrix} - \bar{\jmath}\begin{vmatrix} 1 & 0 \\ 2 & -2 \end{vmatrix} + \bar{k}\begin{vmatrix} 1 & -1 \\ 2 & 2 \end{vmatrix}$

$\therefore L.H.S. = \bar{\imath}[2 - 0] - \bar{\jmath}[-2 - 0] + \bar{k}[2 + 2]$

$\boxed{\therefore \boldsymbol{L.H.S. = 2\bar{\imath} + 2\bar{\jmath} + 4\bar{k}}}$

Now,

$\bar{a} \times \bar{b} = \begin{vmatrix} \bar{\imath} & \bar{\jmath} & \bar{k} \\ 1 & -1 & 0 \\ 0 & 2 & 1 \end{vmatrix}$

$\therefore \bar{a} \times \bar{b} = \bar{\imath}\begin{vmatrix} -1 & 0 \\ 2 & 1 \end{vmatrix} - \bar{\jmath}\begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} + \bar{k}\begin{vmatrix} 1 & -1 \\ 0 & 2 \end{vmatrix}$

$\therefore \bar{a} \times \bar{b} = \bar{\imath}[-1 - 0] - \bar{\jmath}[1 - 0] + \bar{k}[2 - 0]$

$\boxed{\therefore \boldsymbol{\bar{a} \times \bar{b} = -\bar{\imath} - \bar{\jmath} + 2\bar{k}}}$

$\bar{a} \times \bar{c} = \begin{vmatrix} \bar{\imath} & \bar{\jmath} & \bar{k} \\ 1 & -1 & 0 \\ 2 & 0 & -3 \end{vmatrix}$

$\therefore \bar{a} \times \bar{c} = \bar{\imath}\begin{vmatrix} -1 & 0 \\ 0 & -3 \end{vmatrix} - \bar{\jmath}\begin{vmatrix} 1 & 0 \\ 2 & -3 \end{vmatrix} + \bar{k}\begin{vmatrix} 1 & -1 \\ 2 & 0 \end{vmatrix}$

$\therefore \bar{a} \times \bar{c} = \bar{\imath}[3 - 0] - \bar{\jmath}[-3 - 0] + \bar{k}[0 + 2]$

$\boxed{\therefore \boldsymbol{\bar{a} \times \bar{c} = 3\bar{\imath} + 3\bar{\jmath} + 2\bar{k}}}$

$R.H.S. = (\bar{a} \times \bar{b}) + (\bar{a} \times \bar{c})$

$\therefore R.H.S. = -\bar{\imath} - \bar{\jmath} + 2\bar{k} + 3\bar{\imath} + 3\bar{\jmath} + 2\bar{k}$

$\therefore \boldsymbol{R.H.S. = 2\bar{\imath} + 2\bar{\jmath} + 4\bar{k}}$

$\therefore L.H.S. = R.H.S.$

$\boxed{\boldsymbol{\bar{a} \times (\bar{b} + \bar{c}) = (\bar{a} \times \bar{b}) + (\bar{a} \times \bar{c})}}$

- **Homework:**

1. **If $\bar{a} = 2\bar{\imath} + \bar{\jmath} - 3\bar{k}, \bar{b} = \bar{\imath} - 2\bar{\jmath} + \bar{k}$ & $\bar{c} = 2\bar{\imath} + 2\bar{\jmath} + 3\bar{k}$ find vector $(\bar{a} \times \bar{b}) \times \bar{c}$ [Ans: $-5\bar{\imath} + 5\bar{\jmath}$]**
2. **Find $\bar{a} \times \bar{b}$ if $\bar{a} = 2\bar{\imath} + \bar{k}, \bar{b} = \bar{\imath} + \bar{\jmath} + \bar{k}$ $\langle Ans: -\bar{\imath} - \bar{\jmath} + 2\bar{k}\rangle$**
3. **Given $\bar{a} = (\bar{\imath} + 3\bar{\jmath} - 2\bar{k}) \times (-\bar{\imath} + 3\bar{k})$ find the magnitude of $\bar{a}$. $\langle Ans: \sqrt{91}\rangle$**

- **Sine angle between two vectors:**

Sine angle between two vectors $\vec{a}$ and $\vec{b}$ is given by formula,

$$\boxed{\theta = \sin^{-1}\left[\frac{|\vec{a}\times\vec{b}|}{|\vec{a}|\,|\vec{b}|}\right]}$$

- **Example:**

1. Find sine of the angle between the vectors $\vec{a}$ and $\vec{b}$, $\bar{a} = \bar{\imath} - 2\bar{k}, \bar{b} = \bar{\jmath} - 4\bar{k}$ [S-24 4M]

Sol: Consider,

$\bar{a} = \bar{\imath} + 0\bar{\jmath} - 2\bar{k}$

$\bar{b} = 0\bar{\imath} + \bar{\jmath} - 4\bar{k}$

Sine angle between two vectors $\vec{a}$ and $\vec{b}$ is given by formula,

$$\boxed{\theta = \sin^{-1}\left[\frac{|\vec{a}\times\vec{b}|}{|\vec{a}|\,|\vec{b}|}\right]} \quad \ldots\ldots\ldots 1)$$

$$\bar{a}\times\bar{b} = \begin{vmatrix} \bar{\imath} & \bar{\jmath} & \bar{k} \\ 1 & 0 & -2 \\ 0 & 1 & -4 \end{vmatrix}$$

$$\therefore \bar{a}\times\bar{b} = \bar{\imath}\begin{vmatrix} 0 & -2 \\ 1 & -4 \end{vmatrix} - \bar{\jmath}\begin{vmatrix} 1 & -2 \\ 0 & -4 \end{vmatrix} + \bar{k}\begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix}$$

$$\therefore \bar{a}\times\bar{b} = \bar{\imath}[0+2] - \bar{\jmath}[-4-0] + \bar{k}[1-0]$$

$$\boxed{\therefore \bar{a}\times\bar{b} = 2\bar{\imath} + 4\bar{\jmath} + \bar{k}}$$

$$|\bar{a}\times\bar{b}| = \sqrt{(2)^2 + (4)^2 + (1)^2}$$

$$\boxed{\therefore |\bar{a}\times\bar{b}| = \sqrt{21}}$$

Now,

$$|\bar{a}| = \sqrt{(1)^2 + (0)^2 + (-2)^2}$$

$$\boxed{\therefore |\bar{a}| = \sqrt{5}}$$

And

$$|\bar{b}| = \sqrt{(0)^2 + (1)^2 + (-4)^2}$$

$$\boxed{\therefore |\bar{b}| = \sqrt{17}}$$

Equation 1) becomes;

$$\theta = \sin^{-1}\left[\frac{\sqrt{21}}{\sqrt{5}\sqrt{17}}\right]$$

$$\boxed{\therefore \theta = 29.8^0}$$

2. **Compute the sine angle between two vectors 3i + 4j – k and 2i – j + k.**

Sol: Consider,

$\bar{a} = 3\bar{\imath} + 4\bar{\jmath} - \bar{k}$

$\bar{b} = 2\bar{\imath} - \bar{\jmath} + \bar{k}$

Sine angle between two vectors $\vec{a}$ and $\vec{b}$ is given by formula,

$$\boxed{\theta = \sin^{-1}\left[\frac{|\vec{a} \times \vec{b}|}{|\vec{a}|\,|\vec{b}|}\right]} \quad \ldots\ldots\ldots 1)$$

$$\bar{a} \times \bar{b} = \begin{vmatrix} \bar{\imath} & \bar{\jmath} & \bar{k} \\ 3 & 4 & -1 \\ 2 & -1 & 1 \end{vmatrix}$$

$$\therefore \bar{a} \times \bar{b} = \bar{\imath}\begin{vmatrix} 4 & -1 \\ -1 & 1 \end{vmatrix} - \bar{\jmath}\begin{vmatrix} 3 & -1 \\ 2 & 1 \end{vmatrix} + \bar{k}\begin{vmatrix} 3 & 4 \\ 2 & -1 \end{vmatrix}$$

$$\therefore \bar{a} \times \bar{b} = \bar{\imath}[4 - 1] - \bar{\jmath}[3 + 2] + \bar{k}[-3 - 8]$$

$$\boxed{\therefore \bar{a} \times \bar{b} = 3\bar{\imath} - 5\bar{\jmath} - 11\bar{k}}$$

$$|\bar{a} \times \bar{b}| = \sqrt{(3)^2 + (-5)^2 + (-11)^2}$$

$$\boxed{\therefore |\bar{a} \times \bar{b}| = \sqrt{155}}$$

Now, $|\bar{a}| = \sqrt{(3)^2 + (4)^2 + (-1)^2}$

$$\boxed{\therefore |\bar{a}| = \sqrt{26}}$$

$$|\bar{b}| = \sqrt{(2)^2 + (-1)^2 + (1)^2}$$

$$\boxed{\therefore |\bar{b}| = \sqrt{6}}$$

Equation 1) becomes,

$$\theta = \sin^{-1}\left[\frac{\sqrt{155}}{\sqrt{26}\sqrt{6}}\right]$$

$$\boxed{\therefore \theta = 86.47^0}$$

❖ <u>**Homework:**</u>

3. **Find the sine angle between the vectors** $\bar{a} = -4\bar{\imath} + 2\bar{\jmath} + \bar{k}$ & $\bar{b} = 4\bar{\imath} + 3\bar{k}$ $[Ans: 55.77^0]$

4. **Find the sine angle between the vectors** $\bar{a} = 2\bar{\imath} + 3\bar{\jmath} - \bar{k}$ & $\bar{b} = \bar{\imath} - 3\bar{\jmath} + 5\bar{k}$ $[Ans: 57.4^0]$

UNIT 3

VECTORS AND TENSORS (14 MARKS)

Topic Content: 3.1 Introduction, Definition of scalar and vector quantity, Representation of vector, Magnitude of vector, Component of vector, Direction ratio, Direction cosines

3.2 Types of vectors: Zero vector, Unit vector, Position vector, Equal vector, Negative vector. Parallel vector, Co-initial vector, collinear vector

3.3 Algebra of vectors: Addition of vectors, Triangle law of vectors addition, Parallelogram law of Vectors addition, Subtraction of vectors, Multiplication of vectors by scalar

3.4 Product of two vectors: Scalar (dot) product of two vectors, Projection of one vector on another vector, Angle between two vectors using scalar(dot) product, Properties of scalar(dot) product

3.5 Vector (cross) product of two vectors, Angle between two vectors using vector (cross) product, Properties of vector (cross) product

3.6 Scalar triple product of vectors

3.7 Tensor: Definition of tensors, Types of tensors, Rank of tensors, Algebra of tensors

Course Outcome: After completion of this course, students will be able to

CO3: Build programs to implement basic operations based on vectors and tensors.

- **Scalar Triple Product of Vectors:**

Let $\vec{a}, \vec{b}$ and $\vec{c}$ are three vectors then scalar triple product is given as

$\boxed{\vec{a}.(\vec{b} \times \vec{c})}$

- **Examples:**

7. Find scalar triple product of the vector $\bar{a} = \bar{\imath} + \bar{\jmath} + \bar{k}, \bar{b} = 2\bar{\imath} + 4\bar{\jmath} - 5\bar{k}$ & $\bar{c} = 2\bar{\imath} + 2\bar{\jmath} + 3\bar{k}$ [W-23 4M]

Sol: Given,

$\bar{a} = \bar{\imath} + \bar{\jmath} + \bar{k},$

$\bar{b} = 2\bar{\imath} + 4\bar{\jmath} - 5\bar{k}$

$\bar{c} = 2\bar{\imath} + 2\bar{\jmath} + 3\bar{k}$

Scalar Triple Product is given by $\vec{a}.(\vec{b} \times \vec{c})$ ………… 1)

$$\therefore \bar{b} \times \bar{c} = \begin{vmatrix} \bar{\imath} & \bar{\jmath} & \bar{k} \\ 2 & 4 & -5 \\ 2 & 2 & 3 \end{vmatrix}$$

$$\therefore \bar{b} \times \bar{c} = \bar{\imath}\begin{vmatrix} 4 & -5 \\ 2 & 3 \end{vmatrix} - \bar{\jmath}\begin{vmatrix} 2 & -5 \\ 2 & 3 \end{vmatrix} + \bar{k}\begin{vmatrix} 2 & 4 \\ 2 & 2 \end{vmatrix}$$

$$\therefore \bar{b} \times \bar{c} = \bar{\imath}[12 + 10] - \bar{\jmath}[6 + 10] + \bar{k}[4 - 8]$$

$$\boxed{\therefore \bar{b} \times \bar{c} = 22\bar{\imath} - 16\bar{\jmath} - 4\bar{k}}$$

Now,

$\bar{a}.(\bar{b}\times\bar{c}) = (\bar{\imath}+\bar{\jmath}+\bar{k}).(22\bar{\imath}-16\bar{\jmath}-4\bar{k})$

$\therefore \bar{a}.(\bar{b}\times\bar{c}) = 1.22+1.(-16)+1.(-4)$

$\therefore \bar{a}.(\bar{b}\times\bar{c}) = 22-16-4$

$\boxed{\therefore \bar{a}.(\bar{b}\times\bar{c}) = 2}$

8. Find scalar triple product of the vector $\bar{a}=\bar{\imath}-2\bar{\jmath}+3\bar{k}, \bar{b}=2\bar{\imath}+\bar{\jmath}-\bar{k}$ & $\bar{c}=\bar{\jmath}+\bar{k}$

Sol: Given,

$\bar{a}=\bar{\imath}-2\bar{\jmath}+3\bar{k},$

$\bar{b}=2\bar{\imath}+\bar{\jmath}-\bar{k}$

$\bar{c}=0\bar{\imath}+\bar{\jmath}+\bar{k}$

Scalar Triple Product is given by $\vec{a}.(\vec{b}\times\vec{c})$ ……………1)

$$\therefore \bar{b}\times\bar{c} = \begin{vmatrix} \bar{\imath} & \bar{\jmath} & \bar{k} \\ 2 & 1 & -1 \\ 0 & 1 & 1 \end{vmatrix}$$

$$\therefore \bar{b}\times\bar{c} = \bar{\imath}\begin{vmatrix} 1 & -1 \\ 1 & 1 \end{vmatrix} - \bar{\jmath}\begin{vmatrix} 2 & -1 \\ 0 & 1 \end{vmatrix} + \bar{k}\begin{vmatrix} 2 & 1 \\ 0 & 1 \end{vmatrix}$$

$\therefore \bar{b}\times\bar{c} = \bar{\imath}[1+1]-\bar{\jmath}[2-0]+\bar{k}[2-0]$

$\boxed{\therefore \bar{b}\times\bar{c} = 2\bar{\imath}-2\bar{\jmath}+2\bar{k}}$

Now,

$\bar{a}.(\bar{b}\times\bar{c}) = (\bar{\imath}-2\bar{\jmath}+3\bar{k}).(2\bar{\imath}-2\bar{\jmath}+2\bar{k})$

$\therefore \bar{a}.(\bar{b}\times\bar{c}) = 1.2+(-2).(-2)+3.2$

$\therefore \bar{a}.(\bar{b}\times\bar{c}) = 2+4+6$

$\boxed{\therefore \bar{a}.(\bar{b}\times\bar{c}) = 12}$

9. Find a vector of magnitude $\sqrt{7}$ units & perpendicular to the vectors $\bar{a}=2\bar{\imath}+\bar{\jmath}-3\bar{k}, \bar{b}=\bar{\imath}-2\bar{\jmath}+\bar{k}$

Sol: Given,

$\bar{a}=2\bar{\imath}+\bar{\jmath}-3\bar{k},$

$\bar{b}=\bar{\imath}-2\bar{\jmath}+\bar{k}$

A vector of magnitude x units and perpendicular to vectors $\bar{a}$ & $\bar{b}$ is given by,

$$\bar{v} = \frac{\bar{a}\times\bar{b}}{|\bar{a}\times\bar{b}|}x \text{ ………1)}$$

$$\therefore \bar{a}\times\bar{b} = \begin{vmatrix} \bar{\imath} & \bar{\jmath} & \bar{k} \\ 2 & 1 & -3 \\ 1 & -2 & 1 \end{vmatrix}$$

$$\therefore \bar{a}\times\bar{b} = \bar{\imath}\begin{vmatrix} 1 & -3 \\ -2 & 1 \end{vmatrix} - \bar{\jmath}\begin{vmatrix} 2 & -3 \\ 1 & 1 \end{vmatrix} + \bar{k}\begin{vmatrix} 2 & 1 \\ 1 & -2 \end{vmatrix}$$

$\therefore \bar{a}\times\bar{b} = \bar{\imath}[1-6]-\bar{\jmath}[2+3]+\bar{k}[-4-1]$

$\boxed{\therefore \bar{a} \times \bar{b} = -5\bar{\imath} - 5\bar{\jmath} - 5\bar{k}}$

Now,

$|\bar{a} \times \bar{b}| = \sqrt{(-5)^2 + (-5)^2 + (-5)^2}$

$\boxed{\therefore |\bar{a} \times \bar{b}| = 5\sqrt{3}}$

Magnitude of vector is $x = \sqrt{7}$

Equation 1) becomes.

$$\bar{v} = \frac{-5\bar{\imath} - 5\bar{\jmath} - 5\bar{k}}{5\sqrt{3}}(\sqrt{7})$$

$$\boxed{\therefore \bar{v} = -\frac{\sqrt{7}}{\sqrt{3}}\bar{\imath} - \frac{\sqrt{7}}{\sqrt{3}}\bar{\jmath} - \frac{\sqrt{7}}{\sqrt{3}}\bar{k}}$$

- **Note: Two vectors are said to orthogonal (Perpendicular) to each other iff their dot product is zero.**

10. Find a vector of magnitude 7 units & perpendicular to the vectors $\bar{a} = 2\bar{\imath} - 2\bar{\jmath} + 6\bar{k}, \bar{b} = \bar{\imath} + \bar{\jmath} - \bar{k}$

Sol: Given,

$\bar{a} = 2\bar{\imath} - 2\bar{\jmath} + 6\bar{k}$,

$\bar{b} = \bar{\imath} + \bar{\jmath} - \bar{k}$

A vector of magnitude x units and perpendicular to vectors $\bar{a}$ & $\bar{b}$ is given by,

$$\bar{v} = \frac{\bar{a} \times \bar{b}}{|\bar{a} \times \bar{b}|}x \ldots\ldots\ldots 1)$$

$$\therefore \bar{a} \times \bar{b} = \begin{vmatrix} \bar{\imath} & \bar{\jmath} & \bar{k} \\ 2 & -2 & 6 \\ 1 & 1 & -1 \end{vmatrix}$$

$$\therefore \bar{a} \times \bar{b} = \bar{\imath}\begin{vmatrix} -2 & 6 \\ 1 & -1 \end{vmatrix} - \bar{\jmath}\begin{vmatrix} 2 & 6 \\ 1 & -1 \end{vmatrix} + \bar{k}\begin{vmatrix} 2 & -2 \\ 1 & 1 \end{vmatrix}$$

$$\therefore \bar{a} \times \bar{b} = \bar{\imath}[2 - 6] - \bar{\jmath}[-2 - 6] + \bar{k}[2 + 2]$$

$\boxed{\therefore \bar{a} \times \bar{b} = -4\bar{\imath} + 8\bar{\jmath} + 4\bar{k}}$

Now,

$|\bar{a} \times \bar{b}| = \sqrt{(-4)^2 + (8)^2 + (4)^2}$

$\boxed{\therefore |\bar{a} \times \bar{b}| = 4\sqrt{6}}$

Magnitude of vector is $x = 7$

Equation 1) becomes.

$$\bar{v} = \frac{-4\bar{\imath} + 8\bar{\jmath} + 4\bar{k}}{4\sqrt{6}}(7)$$

$$\boxed{\therefore \bar{v} = -\frac{7}{\sqrt{6}}\bar{\imath} + \frac{14}{\sqrt{6}}\bar{\jmath} + \frac{7}{\sqrt{6}}\bar{k}}$$

- **<u>Homework:</u>**

1. Find a vector of magnitude 7 units & perpendicular to the vectors $\bar{a} = 2\bar{\imath} - 3\bar{\jmath} + 6\bar{k}, \bar{b} = \bar{\imath} + \bar{\jmath} - \bar{k}$

11. Are the four points $A(1,-1,1), B(-1,1,1), C(1,1,1), D(2,-3,4)$ coplanar?

Justify your answer. [S-24 4 M]

Sol: Four points are coplanar if the volume of the parallelepiped formed by the vectors representing three edges of the tetrahedron formed by the four points is zero.

Consider,

$A(1,-1,1) \rightarrow \bar{a} = \bar{\imath} - \bar{\jmath} + \bar{k}$

$B(-1,1,1) \rightarrow \bar{b} = -\bar{\imath} + \bar{\jmath} + \bar{k}$

$C(1,1,1) \rightarrow \bar{c} = \bar{\imath} + \bar{\jmath} + \bar{k}$

$D(2,-3,4) \rightarrow \bar{c} = 2\bar{\imath} - 3\bar{\jmath} + 4\bar{k}$

$\overline{AB} = \bar{b} - \bar{a} = -\bar{\imath} + \bar{\jmath} + \bar{k} - \bar{\imath} + \bar{\jmath} - \bar{k}$

$\boxed{\therefore \overline{AB} = -2\bar{\imath} + 2\bar{\jmath} + 0\bar{k}}$

$\overline{AC} = \bar{c} - \bar{a} = \bar{\imath} + \bar{\jmath} + \bar{k} - \bar{\imath} + \bar{\jmath} - \bar{k}$

$\boxed{\therefore \overline{AC} = 0\bar{\imath} + 2\bar{\jmath} + 0\bar{k}}$

$\overline{AD} = \bar{d} - \bar{a} = 2\bar{\imath} - 3\bar{\jmath} + 4\bar{k} - \bar{\imath} + \bar{\jmath} - \bar{k}$

$\boxed{\therefore \overline{AD} = \bar{\imath} - 2\bar{\jmath} + 3\bar{k}}$

Four points are coplanar if $\overline{AB}.(\overline{AC} \times \overline{AD}) = 0 \ldots\ldots\ldots 1)$

$$\therefore \overline{AC} \times \overline{AD} = \begin{vmatrix} \bar{\imath} & \bar{\jmath} & \bar{k} \\ 0 & 2 & 0 \\ 1 & -2 & 3 \end{vmatrix}$$

$$\therefore \overline{AC} \times \overline{AD} = \bar{\imath}\begin{vmatrix} 2 & 0 \\ -2 & 3 \end{vmatrix} - \bar{\jmath}\begin{vmatrix} 0 & 0 \\ 1 & 3 \end{vmatrix} + \bar{k}\begin{vmatrix} 0 & 2 \\ 1 & -2 \end{vmatrix}$$

$\therefore \overline{AC} \times \overline{AD} = \bar{\imath}[6-0] - \bar{\jmath}[0-0] + \bar{k}[0-2]$

$\boxed{\therefore \overline{AC} \times \overline{AD} = 6\bar{\imath} - 0\bar{\jmath} - 2\bar{k}}$

Now,

$\overline{AB}.(\overline{AC} \times \overline{AD}) = (-2\bar{\imath} + 2\bar{\jmath} + 0\bar{k}).(6\bar{\imath} - 0\bar{\jmath} - 2\bar{k})$

$\therefore \overline{AB}.(\overline{AC} \times \overline{AD}) = -2.6 + 2.0 + 0.(-2)$

$\therefore \overline{AB}.(\overline{AC} \times \overline{AD}) = -12 + 0 + 0$

$\boxed{\therefore \overline{AB}.(\overline{AC} \times \overline{AD}) = -12 \neq 0}$

Four points are not coplanar