UNIT 4

NUMERICAL DIFFERENTIATION AND INTEGRATION (12 MARKS)

Topic Content:

4.1 Introduction to numerical differentiation and integration
4.2 Derivative using forward and backward interpolation
4.3 Numerical integration using Trapezoidal rule
4.4 Numerical integration using Simpson's one third rule
4.5 Numerical integration using Simpson's three eight rule

MR SUDHIR S DESAI

Course Outcome: After completion of this course, students will be able to

CO4: Evaluate numerical differentiation and integration functions.

❖ Introduction.

The process of evaluating a definite integral from a set of tabulated values of the integrand $f(x)$ is called numerical integration. This process when applied to a function of a single variable, is known as quadrature.

The problem of numerical integration, like that of numerical differentiation, is solved by representing $f(x)$ by an interpolation formula and then integrating it between the given limits. In this way, we can derive quadrature formulae for approximate integration of a function defined by a set of numerical values only.

❖ Trapezoidal Rule: 4M/6M Que. 5

$$\int_a^b f(x)\,dx = \frac{h}{2}[(y_0 + y_n) + 2(y_1 + y_2 + y_3 + \cdots + y_{n-1})]$$

Where $h = \frac{b-a}{n}$, n = no. of subintervals

❖ Examples:

1. **Evaluate $\int_2^7 \frac{1}{x}\,dx$ using trapezoidal rule & by dividing the interval $[2, 7]$ into five equal sub-intervals.**

Sol: Consider,

$$I = \int_2^7 \frac{1}{x}\,dx$$

Here,

$a = 2, b = 7, n = 5, h = \frac{b-a}{n} = \frac{7-2}{5} \rightarrow$ $h = 1$

x	2	3	4	5	6	7
$y = f(x) = \frac{1}{x}$	0.5	0.3333	0.25	0.2	0.1667	0.1429
	y_0	y_1	y_2	y_3	y_4	y_n

By Trapezoidal Rule.

$$\int_a^b f(x)\,dx = \frac{h}{2}[(y_0 + y_n) + 2(y_1 + y_2 + y_3 + \cdots + y_{n-1})]$$

$$\therefore \int_2^7 \frac{1}{x}\,dx = \frac{1}{2}[(0.5 + 0.1429) + 2(0.3333 + 0.25 + 0.2 + 0.1667)]$$

$$\therefore \int_2^7 \frac{1}{x}\,dx = 1.2715$$

2. Evaluate $\int_0^1 \frac{1}{1+x^2} dx$ by dividing the interval in six subintervals by Trapezoidal rule.

Sol: Consider,

$$I = \int_0^1 \frac{1}{1+x^2} dx$$

Here,

$a = 0, b = 1, n = 6, h = \frac{b-a}{n} = \frac{1-0}{6} \to \boxed{h = 0.1667}$

x	0	0.1667	0.3334	0.5001	0.6668	0.8335	1
$y = f(x) = \frac{1}{1+x^2}$	1	0.9730	0.9	0.7999	0.6922	0.5901	0.5
	y_0	y_1	y_2	y_3	y_4	y_5	y_n

By Trapezoidal Rule.

$$\int_a^b f(x)\, dx = \frac{h}{2}[(y_0 + y_n) + 2(y_1 + y_2 + y_3 + \cdots + y_{n-1})]$$

$$\therefore \int_0^1 \frac{1}{1+x^2} dx = \frac{0.6667}{2}[(1 + 0.5) + 2(0.9730 + 0.9 + 0.7999 + 0.6922 + 0.5901)]$$

$$\boxed{\therefore \int_0^1 \frac{1}{1+x^2} dx = 0.7844}$$

3. Evaluate by Trapezoidal rule $\int_0^5 \frac{1}{4x+5} dx$ dividing the range into 10 equal parts.

Sol: Consider,

$$I = \int_0^5 \frac{1}{4x+5} dx$$

Here,

$a = 0, b = 5, n = 10, h = \frac{b-a}{n} = \frac{5-0}{10} \to \boxed{h = 0.5}$

x	0	0.5	1	1.5	2	2.5	3	3.5	4	4.5	5
$y = f(x) = \frac{1}{4x+5}$	0.2	0.1429	0.1111	0.0909	0.0769	0.0667	0.0588	0.0526	0.0476	0.0435	0.04
	y_0	y_1	y_2	y_3	y_4	y_5	y_6	y_7	y_8	y_9	y_n

By Trapezoidal Rule.

$$\int_a^b f(x)\, dx = \frac{h}{2}[(y_0 + y_n) + 2(y_1 + y_2 + y_3 + \cdots + y_{n-1})]$$

$$\therefore \int_0^5 \frac{1}{4x+5} dx = \frac{0.5}{2}[(0.2 + 0.04) + 2(0.1429 + 0.1111 + 0.0909 + 0.0769 + 0.0667 + 0.0588 + 0.0526 + 0.0476 + 0.0435)]$$

$$\boxed{\therefore \int_0^5 \frac{1}{4x+5} dx = 0.4055}$$

- **Homework:**

1. **Use trapezoidal rule to evaluate $\int_0^1 x^3\,dx$ considering five sub-intervals. (Ans: 0.26)**
2. **Evaluate $\int_1^2 \frac{dx}{x}$ by using Trapezoidal rule $h = 0.25$. (Ans: 0.697)**
3. **Evaluate $\int_0^1 e^x\,dx$ by using Trapezoidal rule with subintervals $n = 10$. (Ans: 1.7196)**
4. **Using Trapezoidal rule evaluate $\int_0^6 f(x)\,dx$ from the following data. Take n = 6. (Ans: 2.5213)**

x	0	1	2	3	4	5	6
f(x)	1	0.5	0.3333	0.25	0.2	0.6666	0.1428

5. **Use Trapezoidal rule to evaluate $\int_0^2 (1+x^3)\,dx$ with $h = \frac{1}{2}$. (Ans: 6.25)**
6. **Evaluate $\int_0^4 (1+x^2)\,dx$ by using Trapezoidal rule taking $h = 1$. (Ans: 71)**
7. **Evaluate $\int_0^6 \frac{1}{1+x^2}\,dx$ by dividing the interval in six subintervals by Trapezoidal rule. (Ans: 1.4108)**
8. **Evaluate $\int_0^2 e^{x^2}\,dx$ by using Trapezoidal rule with subintervals $n = 10$. (Ans: 17.0621)**
9. **Use trapezoidal rule to evaluate $\int_0^1 \frac{1}{1+x}\,dx$ considering five sub-intervals.**

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CO4: Evaluate numerical differentiation and integration functions.

❖ **Simpsons $\frac{1}{3}^{rd}$ Rule: 4M/6M Que. 5**

$$\int_a^b f(x)\,dx = \frac{h}{3}[(y_0 + y_n) + 4(y_1 + y_3 + \cdots + y_{n-1}) + 2(y_2 + y_4 + \cdots + y_{n-2})]$$

Where $h = \frac{b-a}{n}$, n = no. of subintervals

❖ **Examples:**

4. **Evaluate $\int_0^2 \sqrt{x}\,dx$ using Simpson's one third rule & by dividing the interval $(0,2)$ into four sub-intervals.**

Sol: Consider,

$$I = \int_0^2 \sqrt{x}\,dx$$

Here,

$a = 0, b = 2, n = 4, h = \frac{b-a}{n} = \frac{2-0}{4} \to$ $\boxed{h = 0.5}$

x	0	0.5	1	1.5	2
$y = f(x) = \sqrt{x}$	0	0.7071	1	1.2247	1.4142
	y_0	y_1	y_2	y_3	y_n

By Simpson's $\frac{1}{3}^{rd}$ Rule.

$$\int_a^b f(x)\,dx = \frac{h}{3}[(y_0 + y_n) + 4(y_1 + y_3 + \cdots + y_{n-1}) + 2(y_2 + y_4 + \cdots + y_{n-2})]$$

$$\therefore \int_0^2 \sqrt{x}\,dx = \frac{0.5}{3}[(0 + 1.4142) + 4(0.7071 + 1.2247) + 2(1)]$$

$$\boxed{\therefore \int_0^2 \sqrt{x}\,dx = 1.8569}$$

5. Evaluate $\int_0^2 e^{-x}\, dx$ using Simpson's one third rule & by dividing the interval $(0, 2)$ into four sub-intervals.

Sol: Consider,

$$I = \int_0^2 e^{-x}\, dx$$

Here,

$a = 0, b = 2, n = 4, h = \frac{b-a}{n} = \frac{2-0}{4} \rightarrow \boxed{h = 0.5}$

x	0	0.5	1	1.5	2
$y = f(x) = e^{-x}$	1	0.6065	0.3679	0.2231	0.1353
	y_0	y_1	y_2	y_3	y_n

By Simpson's $\frac{1}{3}^{rd}$ Rule.

$$\int_a^b f(x)\, dx = \frac{h}{3}[(y_0 + y_n) + 4(y_1 + y_3 + \cdots + y_{n-1}) + 2(y_2 + y_4 + \cdots + y_{n-2})]$$

$$\therefore \int_0^2 e^{-x}\, dx = \frac{0.5}{3}[(1 + 0.1353) + 4(0.6065 + 0.2231) + 2(0.3679)]$$

$$\boxed{\therefore \int_0^2 e^{-x}\, dx = 0.8649}$$

6. Evaluate the integral $\int_0^1 \frac{x^2}{1+x^3}\, dx$ using Simpsons one third rule into 4 equal sub intervals.

Sol: Consider,

$$I = \int_0^1 \frac{x^2}{1 + x^3}\, dx$$

Here,

$a = 0, b = 1, n = 4, h = \frac{b-a}{n} = \frac{1-0}{4} \rightarrow \boxed{h = 0.25}$

x	0	0.25	0.5	0.75	1
$y = f(x) = \frac{x^2}{1 + x^3}$	0	0.0615	0.2222	0.3956	0.5
	y_0	y_1	y_2	y_3	y_n

By Simpson's $\frac{1}{3}^{rd}$ Rule.

$$\int_a^b f(x)\, dx = \frac{h}{3}[(y_0 + y_n) + 4(y_1 + y_3 + \cdots + y_{n-1}) + 2(y_2 + y_4 + \cdots + y_{n-2})]$$

$$\therefore \int_0^1 \frac{x^2}{1 + x^3}\, dx = \frac{0.25}{3}[(0 + 0.5) + 4(0.0615 + 0.3956) + 2(0.2222)]$$

$$\boxed{\therefore \int_0^1 \frac{x^2}{1 + x^3}\, dx = 0.2311}$$

7. Compute the value of $\int_{0.2}^{1.4}(sinx - logx + e^x)\ dx$ using Simpsons one third rule into 6 equal sub intervals.

Sol: Consider,

$$I = \int_{0.2}^{1.4} (sinx - logx + e^x)\ dx$$

Here,

$a = 0.2, b = 1.4, n = 6, h = \frac{b-a}{n} = \frac{1.4-0.2}{6} \rightarrow \boxed{h = 0.2}$

Set calculator to radian & use log as ln

x	0.2	0.4	0.6	0.8	1	1.2	1.4
$y = f(x) = sinx - logx + e^x$	3.0295	2.7975	2.8976	3.1660	3.5597	4.0698	4.4042
	y_0	y_1	y_2	y_3	y_4	y_5	y_n

By Simpson's $\frac{1}{3}^{rd}$ Rule.

$$\int_a^b f(x)\,dx = \frac{h}{3}[(y_0 + y_n) + 4(y_1 + y_3 + \cdots + y_{n-1}) + 2(y_2 + y_4 + \cdots + y_{n-2})]$$

$$\therefore \int_{0.2}^{1.4} (sinx - logx + e^x)\ dx$$

$$= \frac{0.2}{3}[(3.0295 + 4.4042) + 4(2.7975 + 3.1660 + 4.0698) + 2(2.8976 + 3.5597)]$$

$$\therefore \int_{0.2}^{1.4} (sinx - logx + e^x)\ dx = 4.0321$$

➢ **Homework:**

10. Use Simpson's one third rule to evaluate $\int_0^1 x^3\, dx$ considering five sub-intervals.

11. Evaluate $\int_1^2 \frac{dx}{x}$ by using Simpson's one third rule $h = 0.25$.

12. Evaluate $\int_0^1 e^x\, dx$ by using Simpson's one third rule with subintervals $n = 10$.

13. Using Simpson's one third rule evaluate $\int_0^6 f(x)\, dx$ from the following data. Take n = 6.

x	0	1	2	3	4	5	6
f(x)	1	0.5	0.3333	0.25	0.2	0.6666	0.1428

14. **Use Simpson's one third rule to evaluate $\int_0^2 (1 + x^3)\, dx$ with $h = \frac{1}{2}$.**

15. **Evaluate $\int_0^4 (1 + x^2)\, dx$ by using Simpson's one third rule taking $h = 1$.**

16. Evaluate $\int_0^6 \frac{1}{1+x^2}\, dx$ by dividing the interval in six subintervals by Simpson's one third rule.

17. **Evaluate $\int_0^2 e^{x^2}\, dx$ by using Simpson's one third rule with subintervals $n = 10$.**

18. Use Simpson's one third rule to evaluate $\int_0^1 \frac{1}{1+x}\, dx$ considering five sub-intervals.

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Course Outcome: After completion of this course, students will be able to

CO4: **Evaluate numerical differentiation and integration functions.**

❖ **Simpsons $\frac{3}{8}^{th}$ Rule: 4M/6M Que. 5**

$$\int_a^b f(x)\,dx = \frac{3h}{8}[(y_0 + y_n) + 3(y_1 + y_2 + y_4 + y_5 + \cdots + y_{n-1}) + 2(y_3 + y_6 + \cdots + y_{n-3})]$$

Where $h = \frac{b-a}{n}$, n = no. of subintervals

❖ **Examples:**

8. Evaluate $\int_0^{\frac{\pi}{2}} cosx\ dx$ using Simpson's $\left({}^3/_8\right)^{th}$ rule with $n = 8$.

Sol: Consider,

$$I = \int_0^{\frac{\pi}{2}} cosx\ dx$$

Here,

$$a = 0, b = \frac{\pi}{2}, n = 8, h = \frac{b-a}{n} = \frac{\frac{\pi}{2}-0}{8} \to h = \frac{\pi}{16}$$

Set calculator to radian.

x	0	$\frac{\pi}{16}$	$\frac{2\pi}{16}$	$\frac{3\pi}{16}$	$\frac{4\pi}{16}$	$\frac{5\pi}{16}$	$\frac{6\pi}{16}$	$\frac{7\pi}{16}$	$\frac{\pi}{2}$
$y = f(x) = cosx$	1	0.9808	0.9239	0.8315	0.7071	0.5556	0.3827	0.1951	0
	y_0	y_1	y_2	y_3	y_4	y_5	y_6	y_7	y_n

By Simpson's $\frac{3}{8}^{th}$ Rule.

$$\int_a^b f(x)\,dx = \frac{3h}{8}[(y_0 + y_n) + 3(y_1 + y_2 + y_4 + y_5 + \cdots + y_{n-1}) + 2(y_3 + y_6 + \cdots + y_{n-3})]$$

$$\therefore \int_0^{\frac{\pi}{2}} cosx\ dx = \frac{3\,{}^{\pi}/_{16}}{8}[(1 + 0) + 3(0.9808 + 0.9239 + 0.7071 + 0.5556 + 0.1951) + 2(0.8315 + 0.3827)]$$

$$\therefore \int_0^{\frac{\pi}{2}} cosx\ dx = \frac{3\pi}{128}[(1) + 3(3.3625) + 2(1.2142)]$$

$$\therefore \int_0^{\frac{\pi}{2}} cosx\ dx = 0.9952$$

9. **Evaluate $\int_0^1 \frac{1}{1+x^2}dx$ by using Simpson's $\frac{3}{8}$ th rule dividing the interval $[0, 1]$ in six equal parts. Find the approximate value of π.**

Sol: Consider,

$$I = \int_0^1 \frac{1}{1+x^2}dx$$

Here,

$a = 0, b = 1, n = 6, h = \frac{b-a}{n} = \frac{1-0}{6} \to \boxed{h = 0.1667}$

x	0	0.1667	0.3334	0.5001	0.6668	0.8335	1
$y = f(x) = \frac{1}{1+x^2}$	1	0.9730	0.9	0.7999	0.6922	0.5901	0.5
	y_0	y_1	y_2	y_3	y_4	y_5	y_n

By Simpson's $\frac{3}{8}^{th}$ Rule.

$$\int_a^b f(x)\,dx = \frac{3h}{8}[(y_0 + y_n) + 3(y_1 + y_2 + y_4 + y_5 + \cdots + y_{n-1}) + 2(y_3 + y_6 + \cdots + y_{n-3})]$$

$$\therefore \int_0^1 \frac{1}{1+x^2}dx = \frac{3(0.1667)}{8}[(1 + 0.5) + 3(0.973 + 0.9 + 0.6922 + 0.5901) + 2(0.7999)]$$

$$\boxed{\therefore \int_0^1 \frac{1}{1+x^2}dx = 0.7855}$$

We know that, $\int \frac{1}{1+x^2}dx = \tan^{-1}x + C$

$\therefore [\tan^{-1} x]_0^1 = 0.7855$

$\therefore \tan^{-1} 1 - \tan^{-1} 0 = 0.7855$

$\therefore \frac{\pi}{4} - 0 = 0.7855$

$\therefore \frac{\pi}{4} = 0.7855$

$\therefore \pi = 4 \times 0.7855$

$\boxed{\therefore \pi = 3.1420}$

10. **Using Simpson's $\frac{3}{8}^{th}$ rule to find $\int_0^{0.6} e^{-x^2}\,dx$ by taking seven ordinates.**

Sol: Consider,

$$I = \int_0^{0.6} e^{-x^2}\,dx$$

Here,

$a = 0, b = 0.6, n = 6, h = \frac{b-a}{n} = \frac{0.6-0}{6} \to \boxed{h = 0.1}$

x	0	0.1	0.2	0.3	0.4	0.5	0.6
$y = f(x) = e^{-x^2}$	1	0.99	0.9608	0.9139	0.8521	0.7788	0.6977
	y_0	y_1	y_2	y_3	y_4	y_5	y_n

By Simpson's $\frac{3}{8}^{th}$ Rule.

$$\int_a^b f(x)\,dx = \frac{3h}{8}[(y_0 + y_n) + 3(y_1 + y_2 + y_4 + y_5 + \cdots + y_{n-1}) + 2(y_3 + y_6 + \cdots + y_{n-3})]$$

$$\therefore \int_0^{0.6} e^{-x^2}\,dx = \frac{3(0.1)}{8}[(1 + 0.6977) + 3(0.99 + 0.9608 + 0.8521 + 0.7788) + 2(0.9139)]$$

$$\boxed{\therefore \int_0^{0.6} e^{-x^2}\,dx = 0.5351}$$

- **Homework:**

19. Use Simpson's 3/8 th rule to evaluate $\int_0^1 x^3\,dx$ considering five sub-intervals.

20. **Evaluate $\int_1^2 \frac{dx}{x}$ by using Simpson's 3/8 th rule $h = 0.25$.**

21. Evaluate $\int_0^1 e^x\,dx$ by using Simpson's 3/8 th rule with subintervals $n = 10$.

22. Using Simpson's 3/8 th rule evaluate $\int_0^6 f(x)\,dx$ from the following data. Take n = 6.

x	0	1	2	3	4	5	6
f(x)	1	0.5	0.3333	0.25	0.2	0.6666	0.1428

23. **Use Simpson's 3/8 th rule to evaluate $\int_0^2 (1+x^3)\,dx$ with $h = \frac{1}{2}$.**

24. **Evaluate $\int_0^4 (1+x^2)\,dx$ by using Simpson's 3/8 th rule taking $h = 1$.**

25. Evaluate $\int_0^6 \frac{1}{1+x^2}\,dx$ by dividing the interval in six subintervals by Simpson's 3/8 th rule.

26. **Evaluate $\int_0^2 e^{x^2}\,dx$ by using Simpson's 3/8 th rule with subintervals $n = 10$.**

27. Use Simpson's 3/8 th rule to evaluate $\int_0^1 \frac{1}{1+x}\,dx$ considering five sub-intervals.

28. Evaluate $\int_0^{\frac{\pi}{2}} \sqrt{\cos x}\,\,dx$ using Simpson's $\left({}^3/_8\right)^{th}$ rule with $n = 8$.

29. Evaluate $\int_0^{\pi} \sin x\,\,dx$ using Simpson's $\left({}^3/_8\right)^{th}$ rule with $n = 6$.

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Course Outcome: After completion of this course, students will be able to

CO4: Evaluate numerical differentiation and integration functions.

- **Introduction to Numerical Differentiation**

Numerical differentiation is the process of approximating the derivative of a function using discrete data points. Unlike analytical differentiation, which provides an exact formula for the derivative, numerical differentiation allows us to estimate the derivative when the function is unknown or too complicated for an exact solution.

- **First & second derivative using forward interpolation.**

For $x = x_0$

$$\left[\frac{dy}{dx}\right]_{x=x_0} = \frac{1}{h}\left[\Delta y_0 - \frac{1}{2}\Delta^2 y_0 + \frac{1}{3}\Delta^3 y_0 - \frac{1}{4}\Delta^4 y_0 + \cdots\right]$$

$$\left[\frac{d^2y}{dx^2}\right]_{x=x_0} = \frac{1}{h^2}\left[\Delta^2 y_0 - \Delta^3 y_0 + \frac{11}{12}\Delta^4 y_0 - \frac{5}{6}\Delta^5 y_0 + \frac{137}{180}\Delta^6 y_0 \ldots\right]$$

> **Example:**

1. **Find $\frac{dy}{dx}$ at $x = 1.5$ for the following table.**

x	1.5	2.0	2.5	3.0	3.5	4.0
y	3.375	7.0	13.625	24	38.875	59

Solution: Here $x = 1.5$ is near to initial value so we use forward interpolation formula.

x	y	Δy_0	$\Delta^2 y_0$	$\Delta^3 y_0$	$\Delta^4 y_0$	$\Delta^5 y_0$
1.5	**3.375**	**3.625**	**3**	**0.75**	**0**	**0**
2.0	**7.0**	6.625	3.75	0.75	0	
2.5	**13.625**	10.375	4.5	0.75		
3.0	**24**	14.875	5.25			
3.5	**38.875**	20.125				
4.0	**59**					

Using Formula.

$$h = 2.0 - 1.5 = 0.5$$

$$\left[\frac{dy}{dx}\right]_{x=x_0} = \frac{1}{h}\left[\Delta y_0 - \frac{1}{2}\Delta^2 y_0 + \frac{1}{3}\Delta^3 y_0 - \frac{1}{4}\Delta^4 y_0 + \frac{1}{5}\Delta^5 y_0 - \cdots\right]$$

$$\therefore \left[\frac{dy}{dx}\right]_{x=1.5} = \frac{1}{0.5}\left[3.625 - \frac{1}{2}\times 3 + \frac{1}{3}\times 0.75 - \frac{1}{4}\times 0 + \frac{1}{5}\times 0\right]$$

$$\therefore \left[\frac{dy}{dx}\right]_{x=1.5} = 4.75$$

2. **Find $\frac{dy}{dx}$ & $\frac{d^2y}{dx^2}$ at $x = 0$ using suitable interpolation table.**

x	0	1	2	3	4	5
y	4	8	15	7	6	2

Sol: Here $x = 0$ is near to initial value so we use forward interpolation formula.

x	y	Δy_0	$\Delta^2 y_0$	$\Delta^3 y_0$	$\Delta^4 y_0$	$\Delta^5 y_0$
0	4	4	3	-18	40	-70
1	8	7	-15	22	-30	
2	15	-8	7	-10		
3	7	-1	-3			
4	6	-4				
5	2					

Using Formula.

$$h = 1 - 0 = 1$$

$$\left[\frac{dy}{dx}\right]_{x=x_0} = \frac{1}{h}\left[\Delta y_0 - \frac{1}{2}\Delta^2 y_0 + \frac{1}{3}\Delta^3 y_0 - \frac{1}{4}\Delta^4 y_0 + \frac{1}{5}\Delta^5 y_0 - \cdots\right]$$

$$\therefore \left[\frac{dy}{dx}\right]_{x=0} = \frac{1}{1}\left[4 - \frac{1}{2}\times 3 + \frac{1}{3}\times -18 - \frac{1}{4}\times 40 + \frac{1}{5}\times -70\right]$$

$$\therefore \left[\frac{dy}{dx}\right]_{x=0} = -26.5$$

$$\left[\frac{d^2y}{dx^2}\right]_{x=x_0} = \frac{1}{h^2}\left[\Delta^2 y_0 - \Delta^3 y_0 + \frac{11}{12}\Delta^4 y_0 - \frac{5}{6}\Delta^5 y_0 + \frac{137}{180}\Delta^6 y_0 \ldots\right]$$

$$\therefore \left[\frac{d^2y}{dx^2}\right]_{x=0} = \frac{1}{1^2}\left[3 - (-18) + \frac{11}{12}\times 40 - \frac{5}{6}\times -70\right]$$

$$\therefore \left[\frac{d^2y}{dx^2}\right]_{x=0} = 116$$

3. **Find $\frac{dy}{dx}$ & $\frac{d^2y}{dx^2}$ at $x = 1.1$ using suitable interpolation table.**

x	1.0	1.1	1.2	1.3	1.4	1.5	1.6
y	7.989	8.403	8.781	9.129	9.451	9.750	10.031

Sol: Here $x = 1.1$ is near to initial value so we use forward interpolation formula.

x	y	Δy_0	$\Delta^2 y_0$	$\Delta^3 y_0$	$\Delta^4 y_0$	$\Delta^5 y_0$	$\Delta^6 y_0$
1.0	7.989	0.414	-0.036	0.006	-0.002	0.001	0.002
1.1	8.403	0.378	-0.030	0.004	-0.001	0.003	
1.2	8.781	0.348	-0.026	0.003	0.002		
1.3	9.129	0.322	-0.023	0.005			
1.4	9.451	0.299	-0.018				
1.5	9.750	0.281					
1.6	10.031						

Using Formula.

$$h = 1.1 - 1.0 = 0.1$$

$$\left[\frac{dy}{dx}\right]_{x=x_0} = \frac{1}{h}\left[\Delta y_0 - \frac{1}{2}\Delta^2 y_0 + \frac{1}{3}\Delta^3 y_0 - \frac{1}{4}\Delta^4 y_0 + \frac{1}{5}\Delta^5 y_0 - \frac{1}{6}\Delta^6 y_0 \ldots\right]$$

$$\therefore \left[\frac{dy}{dx}\right]_{x=1.1} = \frac{1}{0.1}\left[0.378 - \frac{1}{2}\times -0.030 + \frac{1}{3}\times 0.004 - \frac{1}{4}\times -0.001 + \frac{1}{5}\times 0.003\right]$$

$$\therefore \left[\frac{dy}{dx}\right]_{x=1.1} = 3.9518$$

$$\left[\frac{d^2y}{dx^2}\right]_{x=x_0} = \frac{1}{h^2}\left[\Delta^2 y_0 - \Delta^3 y_0 + \frac{11}{12}\Delta^4 y_0 - \frac{5}{6}\Delta^5 y_0 + \frac{137}{180}\Delta^6 y_0 \ldots\right]$$

$$\therefore \left[\frac{d^2y}{dx^2}\right]_{x=1.1} = \frac{1}{(0.1)^2}\left[-0.030 - 0.004 + \frac{11}{12} \times -0.001 - \frac{5}{6} \times 0.003\right]$$

$$\therefore \left[\frac{d^2y}{dx^2}\right]_{x=1.1} = -3.7417$$

- **Home Work.**

1. **Find $\frac{dy}{dx}$ at $x = 0$ for the following table.**

x	0.0	0.1	0.2	0.3	0.4
Y	1.0000	0.9975	0.9990	0.9776	0.8604

2. **Find $\frac{dy}{dx}$ at $x = 0.1$ for the following table.**

x	0.0	0.1	0.2	0.3	0.4
Y	1.0000	0.9975	0.9990	0.9776	0.8604

3. **Following data gives corresponding values of pressure & volume of super-heated system.**

V	2	4	6	8	10
P	105	42.5	25.3	16.3	13

Find the rate of change of pressure w. r. to volume when V = 2.

Topic Content:

4.1 Introduction to numerical differentiation and integration
4.2 Derivative using forward **and backward interpolation**
4.3 Numerical integration using Trapezoidal rule
4.4 Numerical integration using Simpson's one third rule
4.5 Numerical integration using Simpson's three eight rule

MR SUDHIR S DESAI

Course Outcome: After completion of this course, students will be able to

CO4: Evaluate numerical differentiation and integration functions.

❖ **Introduction to Numerical Differentiation**

Numerical differentiation is the process of approximating the derivative of a function using discrete data points. Unlike analytical differentiation, which provides an exact formula for the derivative, numerical differentiation allows us to estimate the derivative when the function is unknown or too complicated for an exact solution.

❖ **First & second derivative using backward interpolation.**

For $x = x_n$

$$\left[\frac{dy}{dx}\right]_{x=x_n} = \frac{1}{h}\left[\nabla y_n + \frac{1}{2}\nabla^2 y_n + \frac{1}{3}\nabla^3 y_n + \frac{1}{4}\nabla^4 y_n + \cdots\right]$$

$$\left[\frac{d^2y}{dx^2}\right]_{x=x_n} = \frac{1}{h^2}\left[\nabla^2 y_n + \nabla^3 y_n + \frac{11}{12}\Delta^4 y_n + \frac{5}{6}\nabla^5 y_n + \frac{137}{180}\nabla^6 y_n \ldots\right]$$

➢ **Example:**

4. Find $\frac{dy}{dx}$ at $x = 4$ for the following table.

x	1.5	2.0	2.5	3.0	3.5	4.0
y	3.375	7.0	13.625	24	38.875	59

Solution: Here $x = 4$ is near to final value so we use forward interpolation formula.

x	y	∇y_n	$\nabla^2 y_n$	$\nabla^3 y_n$	$\nabla^4 y_n$	$\nabla^5 y_n$
1.5	**3.375**					
2.0	**7.0**	3.625				
2.5	**13.625**	6.625	3			
3.0	**24**	10.375	3.75	0.75		
3.5	**38.875**	14.875	4.5	0.75	0	
4.0	**59**	**20.125**	**5.25**	**0.75**	**0**	**0**

Using Formula.

$$h = 2.0 - 1.5 = 0.5$$

$$\left[\frac{dy}{dx}\right]_{x=x_n} = \frac{1}{h}\left[\nabla y_n + \frac{1}{2}\nabla^2 y_n + \frac{1}{3}\nabla^3 y_n + \frac{1}{4}\nabla^4 y_n + \cdots\right]$$

$$\therefore \left[\frac{dy}{dx}\right]_{x=4.0} = \frac{1}{0.5}\left[20.125 + \frac{1}{2}\times 5.25 + \frac{1}{3}\times 0.75 + \frac{1}{4}\times 0 + \frac{1}{5}\times 0\right]$$

$$\therefore \left[\frac{dy}{dx}\right]_{x=4} = 46$$

5. **Find $\frac{dy}{dx}$ & $\frac{d^2y}{dx^2}$ at $x = 5$ using suitable interpolation table.**

x	0	1	2	3	4	5
y	4	8	15	7	6	2

Sol: Here $x = 4$ is near to final value so we use forward interpolation formula.

x	y	∇y_n	$\nabla^2 y_n$	$\nabla^3 y_n$	$\nabla^4 y_n$	$\nabla^5 y_n$
0	4					
1	8	4				
2	15	7	3			
3	7	-8	-15	-18		
4	6	-1	7	22	40	
5	2	-4	-3	-10	-30	-70

Using Formula.

$$h = 1 - 0 = 1$$

$$\left[\frac{dy}{dx}\right]_{x=x_n} = \frac{1}{h}\left[\nabla y_n + \frac{1}{2}\nabla^2 y_n + \frac{1}{3}\nabla^3 y_n + \frac{1}{4}\nabla^4 y_n + \frac{1}{5}\nabla^5 y_n \ldots\right]$$

$$\therefore \left[\frac{dy}{dx}\right]_{x=5} = \frac{1}{1}\left[-4 + \frac{1}{2}\times -3 + \frac{1}{3}\times -10 + \frac{1}{4}\times -30 + \frac{1}{5}\times -70\right]$$

$$\therefore \left[\frac{dy}{dx}\right]_{x=5} = -30.3333$$

$$\left[\frac{d^2y}{dx^2}\right]_{x=x_n} = \frac{1}{h^2}\left[\nabla^2 y_n + \nabla^3 y_n + \frac{11}{12}\Delta^4 y_n + \frac{5}{6}\nabla^5 y_n + \cdots\right]$$

$$\therefore \left[\frac{d^2y}{dx^2}\right]_{x=5} = \frac{1}{1^2}\left[-3 + (-10) + \frac{11}{12}\times -30 + \frac{5}{6}\times -70\right]$$

$$\therefore \left[\frac{d^2y}{dx^2}\right]_{x=5} = -98.8333$$

6. **Find $\frac{dy}{dx}$ & $\frac{d^2y}{dx^2}$ at $x = 1.6$ using suitable interpolation table.**

x	1.0	1.1	1.2	1.3	1.4	1.5	1.6
y	7.989	8.403	8.781	9.129	9.451	9.750	10.031

Sol: Here $x = 1.6$ is near to final value so we use forward interpolation formula.

x	y	∇y_n	$\nabla^2 y_n$	$\nabla^3 y_n$	$\nabla^4 y_n$	$\nabla^5 y_n$	$\nabla^6 y_n$
1.0	7.989						
1.1	8.403	0.414					
1.2	8.781	0.378	-0.036				
1.3	9.129	0.348	-0.030	0.006			
1.4	9.451	0.322	-0.026	0.004	-0.002		
1.5	9.750	0.299	-0.023	0.003	-0.001	0.001	
1.6	10.031	0.281	-0.018	0.005	0.002	0.003	0.002

Using Formula.

$$h = 1.1 - 1.0 = 0.1$$

$$\left[\frac{dy}{dx}\right]_{x=x_n} = \frac{1}{h}\left[\nabla y_n + \frac{1}{2}\nabla^2 y_n + \frac{1}{3}\nabla^3 y_n + \frac{1}{4}\nabla^4 y_n + \frac{1}{5}\nabla^5 y_n + \frac{1}{6}\nabla^6 y_n \ldots\right]$$

$$\therefore \left[\frac{dy}{dx}\right]_{x=1.6} = \frac{1}{0.1}\left[0.281 + \frac{1}{2}\times -0.018 + \frac{1}{3}\times 0.005 + \frac{1}{4}\times 0.002 + \frac{1}{5}\times 0.003 + \frac{1}{6}\times 0.002\right]$$

$$\therefore \left[\frac{dy}{dx}\right]_{x=1.6} = 2.7510$$

$$\left[\frac{d^2y}{dx^2}\right]_{x=x_n} = \frac{1}{h^2}\left[\nabla^2 y_n + \nabla^3 y_n + \frac{11}{12}\Delta^4 y_n + \frac{5}{6}\nabla^5 y_n + \frac{137}{180}\nabla^6 y_n \dots\right]$$

$$\therefore \left[\frac{d^2y}{dx^2}\right]_{x=1.1} = \frac{1}{(0.1)^2}\left[-0.018 + 0.005 + \frac{11}{12}\times 0.002 + \frac{5}{6}\times 0.003 + \frac{137}{180}\times 0.002\right]$$

$$\boxed{\therefore \left[\frac{d^2y}{dx^2}\right]_{x=1.1} = -0.7144}$$

- **Home Work.**

4. **Find $\frac{dy}{dx}$ at $x = 0.4$ for the following table.**

x	0.0	0.1	0.2	0.3	0.4
Y	1.0000	0.9975	0.9990	0.9776	0.8604

5. **Find $\frac{dy}{dx}$ at $x = 0.3$ for the following table.**

x	0.0	0.1	0.2	0.3	0.4
Y	1.0000	0.9975	0.9990	0.9776	0.8604

6. **Following data gives corresponding values of pressure & volume of super-heated system.**

V	2	4	6	8	10
P	105	42.5	25.3	16.3	13

Find the rate of change of pressure w. r. to volume when V = 10.